On osculation, superosculation and characteristic points

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Dissertation *

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Introduction

Already from the very beginnings, the theory of tangents and osculations in classical differential geometry is an important chapter. This chapter in some sense is even older than differential geometry itself, because constructions of tangents was a problem of ancient Greek mathematicians.

Considering the material of this work in wider treatises of differential geometry and calculus, it is easy to notice some inequality in the discussion of some problems. While the touching of two curves or curves and a surface time to time is examined quite completely, while investigating it also when the touch point is singular, but when it comes to a given type of osculating figure, it tends to be limited to determination of the order of touching, possibly mentioning that the rank can become larger - i.e., superosculation occurs - only at certain points of the given curve or surface. It seems that so far only in the case of special osculating figures, which are uniguely determined, - for line, circle, conic, and so on, in the plane; for line, plane, plane and sphere in the space -, superosculation is investigated for each point of the curve.

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So the idea came to consider more general cases too, when osculating figure is not determined uniquely, and as a special example to consider osculating cylinder for spacial curves.

It turns out, that from such general view looking, mentioned osculating figures correspond to some exceptional case that has very special features. In the same time it was possible to ascertain that the notion of distance, that is used to be applied to characterize order of touching, is not necessaary at all, at least in the case when we exclude from consideration singular points. At last, complete paralelism was discovered between problems: to determine osculating figure for given line, and: to determine characteristic points for one parameter family of figures.

The fact that this parallelism was not previously used seems to be a consequence of two circumstances: In the study of characteristic points, the equation to be consider is with one parameter, against which must be derived, and several unknowns, - against coordinates of points. For osculating figure, equation is given with several unknowns - determiners parameters of surface -, but the only parameter of this equation is set in it only a posteriori, giving coordinates of point as functions of parameter. To research curve or plane which equation has only one coordinate and several parameters seems be undertaking without sense while corresponding problem - to research points which are determined by one equation with several coordinates and one parameter -, as was said, sometimes is put in the basis of the theory of envelopes.

Second case, when it comes to looking at osculating figure and characteristic points at a time, that is, looking at the relationship between the points of spacial curve and its osculating planes, the perfect parallelism encountered there time to time is atributed more or less intuitively to projective duality between point and plane. In fact, not only projective transformations, but in general continuous transformations of points and tangents keep the relation of second order touching between transformed figures of curves and families of planes. This fact is known for a long time, but it seems not to have been followed enough.

This work is generally performed in the spirit of classical differential geometry, which requires all necessary derivatives for all functions, thereby bypassing sigular points. While the assumption about the existence of derivatives for the method used here is fundamental, the exclusion of singular points is made for convenience in order to investigate what happens in general cases. This is done because the study of singularities in any case is impossible already in fairly simple cases, for example, in the problem of envelope of one parameter family of curves for a plane. More close dealing with singularities would require to raise the volume of the work considerably.

In the first chapter of this work, on the basis of consideration, n dimensional or arbitrarily characterized space is put. Order of touching of surface and curve, as well as two curves, is characterized; possiblity to characterize order of touching of some varieties is shown, and it is checked that, determining in the space some metric of general nature, characteristics of order of touching are equivalent with usually used. It follows generalizations of two theorems of seemingly metrical nature.

When considering one-parameter families of surfaces, the already mentioned parallelism with the problem of osculation is encountered and the geometric locations of characteristic points and their properties are considered; also the concept of a supercharacter point and its order are fixed.

In the third paragraph, the criterion for finding the order of solving the system of equations is considered; this is necessary for further conclusions - we could not find such a criterion in the literature.

Next, we study various possibilities for the implementation of superosculations, indicate the method for finding curves with a maximal tangent with a given family of surfaces, and indicate the connection of this problem with singular solutions of differential equations. Simplifications in calculations follow in the study of properties that are invariant with respect to some given transformation group.

In the second chapter, mainly to illustrate the conclusions of the first chapter, some issues related to osculating cylinders of spatial curves of three-dimensional Euclidean space are considered.

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The elements of general theory of osculatation and superosculation

§1. On tangent, osculation and superosculation in a distinct point

Let us put in the basis of our consideration some n dimensional variety, the elements of which we would call points. Let us assume that points of variety may be equiped uniquely with mutually independent coordinates x_i (i=1,2,...,n), and that these coordinates may take all real and complex values. If differences of all coordinates of two points tends to 0, we are to say that the point tends to other point, or , that the points are infinitly close each to other. By giving all points as one parameter t functions

(1) $x_i = x_i$(t) i=1,2,...,n ,

that are defined either for all values of t, or for some interval of values, we determine family of points of n dimensions contained in the variety T, that we are calling curve L. (1) are the parametric equations of the curve; points which coordinates we receive with the expressions (1) for a chosen value t, are points of the curve. Giving one relation between x_i

(2) $g\,(x_1, x_2, ..., x_n\,)$=0

we determine n-1 dimensionl variety V that is contained in the variety T, that we are going to call surface. (2) is the equation of this surface; point which coordiantes obeys the equation of the surface is the point of the surface; surface goes through each of its point; points which coordinates obey severaly equations of surfaces, are the points of intersection of these surfaces. If n=2 the notion of surface is identical with the notion of curve. Considering further families of surfaces S where general surface is depending from N independent paremeters a_j (j=11,2,...,N), let us assume that general surface of family has in correspondence only one system of values of parameter a_j, and that the system of values a_j has one definite surface in correspondence. The curve L is contained in the surface V if all points of curve obey to the equation of the surface.

Example of the notions introduced: n dimensional afine space T, x_i afine co-

ordinates of this space, S - family of curves of second order curves (if n=2), of surfaces (if n=3), hypersufaces (if n>3), respectively.
Higher asumptions on unambiguity of expressions for points and coordinates, as well between surfaces and corresponding parameters, as for opportunity to asign to coordinates x_i and parameters a_j arbitrary values is not necessary, but only serves only for simplification of further conclusions. Leaving them we had to introduce in the formulations certain confinements, or to deal with equations of different type (e.g., in case of homogeneos coordinates). Because esentially the theory to be cosidered shouldn't change considerably, we confine ourselves with the simpest and most transparent (generic) case, when all asumptions are in force.
In order not to repeat further judgements, we perform them for most general case, when n>2. In case n=2 they remain in force, only verbal expressions "surface V", "curve on surface V" should be replaced with "curve V".
Further we use following designations and shortend expressions: the point with coordinates x_i (i=1,...,n) we call point X, as well surface, that has values of parameters a_j (j=1,2,...N) in correspondence, surface A. The fact that function f depends from all or some x_i, and from all or some a_j we are expressing writing f(x, a). f[x(t), a] and f[x, a(t)] shall designate the functions that must be converted to f(x, a) after replacing all x_i, correspondingly all a_j, with some parameter t function $x_i(t)$, or a_j(t) correspondingly.
The derivatives of one parameter t function f(t) after t we denote with the usual symbols f'=$\frac{df}{dt}$, $f^{(k)} = \frac{d^k f}{dt^k}$. Repeatedly we encounter equations that arise equating left side derivative to zero from some given equation f=0 - in such case we say that we derivate equation f=0. If some equation is identically satisfied, all its derivatives are identically satisfied. All functions we encounter below are unambiguous, unless otherwise specified, and continuous; they have continuous derivatives of all orders in all formulae in all possible judgements.

2. We characterize general surface of family S with

(3) f(x, a) = 0.

We determine the curve L with parametric equations

(4) $x_i = x_i(t)$.

Surface A goes through p poins $X_1, X_2, ..., X_p$ of curve L with parametric values $t_1, t_2, ..., t_p$ in corresponce if expressions all at once are satisfied:

$$(5)\quad \left\{ \begin{array}{l} F\,(t_1, a) = 0 \\ F\,(t_2, a) = 0 .....F\,(t_p, a) = 0 \end{array} \right.$$

where

(6) F(t, a) =f [x(t), a].

From further consideration we exclude singular points of surface (3), where all $\frac{\partial f}{\partial x_i}$ vanish, and singular points of curve L, where all $\frac{dx_i}{dt}$ vanish.

Making points $X_1, X_2, ..., X_p$ tend along the curve L to one of its points X_0, characterized by $t = t_0$, using for this case usual judgement[1], we conclude that

$$(7)\quad \left\{ \begin{array}{l} F\,(t_0, a) = 0 \\ F'\,(t_0, a) = 0 .....F^{(p-1)}\,(t_0, a) = 0 \end{array} \right.$$

The fact that expressions (7) hold and besides

(8) $F^{(p)}\,(t_0\,,\,\mathrm{a}) \neq 0$

we are to express in three different ways: a) surface A goes through p infinitaly closed points (of curve L), that coincide with the point X_0; b) the surface A and curve L has in the point X_0tangent of order p-1; c) solving the system of equations of (3) and (4) with respect to unkown x_i and t, just p systems of solutions coincide with the system composed from coordinates of X_0 and t_0, i.e., this system is p-fold solution of the system of equations. The last statement we are to use also in the case when all $\frac{\partial f}{\partial x_i}$ wanish.

If the curve L is given arbitrarily, maximal available value for its general point X is N. Truly, if p=N, putting t_0 in t, system

(9) $\{ \quad \begin{array}{l} F(t,a) = 0 \\ F'(t,a) = 0.....F^{(N-1)}(t,a) = 0 \end{array}$

has N equtions with N unknown a_j. If identically

(10) $\frac{D\left(F,F',...F^{(N-1)}\right)}{D(a_1,a_2,,...,a_N)}$=0

is not true for any values of a_j, system (9) can be solved with respect to parameters a_j, obtaining them as functions of t. In the general case, if all equations of (9) are not linear with respect to all a_j, we obtain several systems of solutions. Surface A corresponding to each such system we are going to call the osculating surface to the curve L at point X. We have come to the known fact (if words *point*, *surface* and *curve* regain their usual meaning): osculating surface, which equation depends from N parameters, with general curve in its general point, has tangent of order N-1.

It is easy to see that the touching of surface and curve doesn't depend from choice of parameter t. More precisely: expressing t as invertible unambiguous function from some other pereameter s, t(s), where value of t, t_0, hasvalue$s = s_0$ in correspondence, function F(t,a) is replaced by some function G(s,a) of parameter s:

G(s,a)=F[t(s), a].

From existence of relationships (7) and (8) follow existence of the same relationship in the point where $s = s_0$ for function G and its derivatives with respect to s, and reversely; values of a_j in both cases should be the same, following, independent from choice of parameters. Really, h-fold derivative of function G with respect to s is expressible as sum of memebers that contain as factors derivatives of F with respect to t up to order h and derivatives of t with respect to s; the only member containing $F^{(h)}$ is $F^{(h)}$(dt/ds)^h. If F and $F^{(h)}$, where h=1,..,p-1, are equal to zero and $F^{(p)}$≠0, when $t = t_0$, these same expressions hold both for quantities G, $G^{(h)}$ and $G^{(p)}$, when $s = s_0$, reversely.

3. The notion of order of touching of surface and curve may be widened firstly in place of one equation (3) taking several equations between x_i, in number m<n:

(11) f_k(x) = 0 k=1, 2, ..., n

They are to characterize in general case n-m dimensional variety V'. Let us exclude from consideration eventual singular cases, when number of dimensions of V' is greater than n - m, and singular points, by requiring in point X_0 the rank of matrix

(12) $\left(\frac{\partial f_k}{\partial x_i}\right)$ $\begin{matrix} k=1,2,...,m \\ i=1,2,...,n \end{matrix}$

to be m. Then we may say: curve L and variety V' really has p-fold touching in point X_0, it has at least p-fold tangent in this point with surfaces that are determined by equations (11) each taken separately, and at least one of these surfaces has just p-fold tangent.

Especially, when m = n - 1, system

(13) f_k(x) = 0 k= 1,2,...,n-1

determines some curve of L'.

Let us determine the most convenient criterion for characterization of touching of two curves. If necessary renumerating coordinates, in accordance with asumption about matrix (12) we may acieve that functional determinant

(14) $\Delta = \frac{D(f_1, f_2, ..., f_{n-1})}{D(x_1, x_2, ..., x_{n-1})}$

is equal to zero neighter in point X_0 nor in close to it, i.e., for points with sufficiently small difference of coordinates from X_0. Then the system may be solved against x_1, x_2, ..., x_{n-1}, receiving them as unambiguous functions of x_n. In turn, expressing x_n as some invertible unique function of parameter s, we receive for by system (13) determined curve L' parametric image of neighborhood of point X_0

(15) x_i= ψ_i(s) i = 1, 2, ..., n

Besides

$\frac{\mathrm{d}\psi_n}{\mathrm{da}} \neq 0$

in the point X_0 and its neighborhood. We determine curve L, using equation of type (4),

(16) x_i = ϕ_i(t) i = 1, 2,...,n

If curves L and L' touch at point X_0, they both pass through this point. So that exist such certain values of s and t, s_0 and t_0 that

ψ_i (s_0) = ψ_i (t_0) = $x_{\mathrm{i}0}$ i = 1, 2,..., n ,

where $x_{\mathrm{i}0}$ are coordinates of the point X_0.

If curve L has at point X_0 just p-fold (p$\geqslant$1) touching with curve L', then according our definition equations are satisfied

(17) f_k [ϕ(t)] = 0 k = 1, 2,...,n-1

as well as their derivatives up to order p, if t = t_0, but at least at one k

(18) $f_k{}^{(p+1)}$ [ϕ (t_0] $\neq$ 0.

Since equations

(19) $f_k(\psi)$ = 0 k = 1, 2,..., n-1

are satisfied identically, arbitrary value of s, $s = s_0$ too, satisfy their derivatives.

Curve L has at point X_0 at least touching of first order; thus

(20) $\sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} \frac{d\phi_i}{dt} = 0$ k = 1, 2 ...,n-1 (if t = t_0)

exist. Deriving equation (19) and putting s = s_0, we receive

(21) $\sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} \frac{d\psi_i}{ds} = 0$ k = 1, 2 ...,n-1 (if s = s_0).

Systems (20) and (21) consist of n-1 linear homogen equations with respect to values of first derivatives of ϕ_i and ψ_i at point X_0. Both systems have the same coefficients, because they are functions of $x_{\mathrm{i}0}$. Since rank of the matrix of these coefficients is n-1, according assumption about determinant (14), it follows that at point X_0 values of first order derivatives of ϕ and ψ are proportional and besides

$\frac{d\phi_n}{dt} \neq 0$

at point X_0 and its neighborhood, because otherwise all derivatives of ϕ_i against t would wanish, and X_0would be singular point of L. Thus, ϕ_n(t) is invertibly unique function in neighborhood of t = t_0 (i.e., for sufficiently small $|t - t_0|$). Equation

ψ_n(s) = ϕ_n(t)

determines s as invertibly unique function of t in neighborhood of corresponding points s_0 and t_0. Inserting this value of s in the parametric equations of curve L', we receive new parametric image, that we can write in this way:

(22) $x_i = \psi_i$(t) ,

besides

ψ_n(t) = ϕ_n(t)

and

$$\left\{ \begin{array}{ll} \psi_i\,(t_0) & = \phi_i\,(t_0) \\ \psi_i'\,(t_0) & = \phi_i'\,(t_0) \end{array} \right. \qquad \text{i = 1, 2,..., n}$$

because marked values of the last row, as we stated, are proportional, and for index value n they are equal.

Let us state, that in case of p-fold touching

$$(23)\ \psi_i{}^{(h)}\ (t_0\,) = \phi_i{}^{(h)}\ (t_0\,) \qquad \begin{array}{c} i = 1, 2, .., n \\ h = 0, 1, 2, .., n \end{array} \qquad \begin{array}{ll} \psi_i{}^{(0)} & = \psi_i \\ \phi_i{}^{(0)} & = \phi_i \end{array}$$

(24) $\psi_i{}^{(p+1)}\ (t_0\,) \neq \phi_i{}^{(p+1)}\ (t_0\,)$ at least for one i.

As we have already seen, conditions (23) are satisfied for the values of h 0 and 1. Assuming that they are satisfied if h $\leq$ r (r $\leq$ p-1), we can say that they are also satisfied in the case of h = r + 1. Determined by (22) functioins ψ_i indentically satisfy equations (19). Expressing that t = t_0 satisfy derivatives of order r+1 of equations (17) and (19), we receive two systems of equations of this kind

$\sum_{i=1}^{n} \frac{\partial f_k}{\partial x_{i0}}\ \phi_i{}^{(r+1)}\ (t_0) + G_k = 0$ k = 1, 2 ...,n-1

$\sum_{i=1}^{n} \frac{\partial f_k}{\partial x_{i0}}\ \psi_i{}^{(r+1)}\ (t_0) + H_k = 0$ k = 1, 2 ...,n-1 .

Quantities G_k and H_k are to be computed in the same way with derivatives of functions ϕ_i, respectively ψ_i, up to order r, if t = t_0, and values of partial equations of functions f_k at point X_0. Subtracting from k-th equation of second order k-th equation of first system, we receive

(25) $\sum_{i=1}^{n} \frac{\partial f_k}{\partial x_{i0}}\ [\psi_i{}^{(r+1)}\ (t_0)$ - $\phi_i{}^{(r+1)}\ (t_0)] = 0$ k = 1, 2 ...,n-1

because according our assumptions all other members mutually disappear. Since determinant of system (25) isn't equal to zero, it follows that (23) holds also in case h = r + 1.

Thus, conditions (23) are satisfied for all values of h from 0 to p. If (24) were satisfied for no i, p+1-fold derivatives of all equations (17) in point X_0 would be satisfied, that would mean at least p+1-fold touching, contradicting hypothesis.

Reversely, if conditions (23) and (24) are satisfied, curve L touches curve L' in point X_0 with coordinates

$x_{i0} = \psi_i\ (t_0\,) = \ \phi_i\ (t_0\,)$

Really, then functions $f_k(\phi)$ and $f_k\ (\psi)$ with their derivatives up to order p have the same values, so that (17) holds (*written with hand*: ... up to p). If (18) were not true, then would follow that system (25) with r=p would satisfy values of brackets, that all are not equal to zero. But it isn't possible, because determinant of system is not equal to zero.

Since conditions (23) and (24) are symmetric with respect to current coordinates of curves L and L' and in place of parameter p they could be formulated with help of s, in case of p-fold touching both curves have symmetric role: curve L' touches curve L with order p at point X_0 too, or, saying otherwise: curves L and L' touch each other at point X_0 with order p.

Collecting the previous we may say:

if two curves have at point X_0 p-fold touching, their parametric equations (16) and (22) may be chosen so that the same parametric value t_0 characterize point X_0 on both curves, and conditions (23) and (24) are satisfied for this value, and reversely.

The last statement allows without difficulty to construct curves L' that have with the given curve L in its given point X_0 is just given order p touching. Current coordinates of curve L' may be taken even in general case as polynoms of order p of parameter t, and in special cases - when no polinom determined by expression (23) would satisfy condition (24) - as polinom of order p+1 of parameter t.

4. As we have seen in the second part of this paragraph, surface A of family S, that at point X_0 osculates curve L, is determined by system (9). If besides values of a_j determined by this system satisfy relationships

$$(26) \qquad \begin{array}{c} F^{(k)}(t,a) = 0 \\ F^{(N+r)}(t,a) \neq 0 \end{array} \qquad \text{k = N, N+1, ..., N+r-1}$$

curve L has with surface A exactly N+r-1-fold touching. Characterizing this, we are going to say that curve L at point X_0 superosculation of order r, or also, that surface superosculates curve with order p.

Inserting given by system (9) values of a_j in conditions (26) and, each of them becomes equation with respect to t. Since, for an arbitrary given curve, the roots of the first equation (26) will not satisfy the second equation, at the points corresponding to these roots, the first order superosculation must occur; higher order superosculation is impossible.

If to contrary curve L lies on one or several (in the finite number) surfaces V, their corresponding values of parameter a_j will satisfy equations (9) and (26) for arbitrary high k. In this case we may speak of infinitely large order of superosculation.

It is not difficult to construct examples with order of superosculation with finite and arbitrary large order either at a separate point, if family S is given, or in each point of given curve L. In the first case it sufices to take curve L that has arbitrary large order p touching with arbitrary curve L' of some surface V. In the second case, we construct firstly second curve L' for each point X of curve L that has exactly p-fold touching with L in this point. Passing through each curve

L' arbitrary surface V, it will have at least p-fold touching with curve L, and we can always achieve that this order is exactly p. In this way we always may attach to each given curve L family of surfaces S of one parameter where for each surface at each point of L it touches with arbitrary large finite order p. Taking family of curves L, that are depending from N-1 parameters, and attaching to each family of surfaces of one argument in the way we just described, we will get family of surfaces with arbitrary large number N of parameters, where each separate surface will touch curve L in each its point with arbitrary large order p.

Naturally the question arises: for a given family S of surfaces V, is it possible to find such curve L having in each point order r superosculation with some of surfaces V, where r⩾1 and finite. This problem we shall consider in paragraph 4, finding before that some other formulation, and finding order of solution of system of equations in order to characterize the result to be obtained.

5. Using notion of order of touching of curves, we may in general way characterize the touching order of any two varieties Q and Q' of dimensions m and m' contained in variety T: at the common point X_0 it is exactly p, because for each curve L of first variety through point X_0 may be found at least one curve L' of second variety, that has p-fold touching with L at point X_0, and besides in this variety exists at least one curve L which touching with L' can't be higher than p. Similarly as in case of two curves, here too coordinates x_i of current point X of both varieties may be expressed as as m, respectively m', parameter functions, since all determining parameters of point of variety are among parameters of second variety. In case of p-fold touching partial derivatives of all x_i against common parameters up to order p at point X_0 has the same values, but starting from p+1-th order derivatives differ at least at pair of values.

Similarly we could seek, by giving some family of varieties S', variety of this family that touches given curve or given variety Q' with possibly high order at its point X. The augment of order of touching can't be characterized by one condition, as it was in case of curve and surface, but with several new conditions. So in general case here should be infinitely many varieties Q of family S' with maximal order of touching. In order to characterize one (or several in finite amount), that could be called osculating variety, we would need to introduce some additional conditions. Such cases are already known in the classical differential geometry of three dimensions in Euclidean space: there is, e.g., simple infinity in case of simple srew lines, which have with given line in given point touching of second order, whereas in the general case none of them has order three.

Especially with these generalizations, we will not deal with, turning attention mainly to points, surfaces and their one parameter families.

6. By binding to V a certain very general metric, we can characterize the order of tangency by comparing an infinitely small distance, as is used in some metric geometries. For this purpose will suffice as a distance ds between two infinitisimally close points X with coordinates x_i - we will write $X(x_i)$ - and $Y(x_i + \mathrm{dx}_i)$ to take some function of x_i and dx_i, that should suit only following

condition: the order of ds for arbitrary dx_i is equal to smallest of orders of dx_i. Otherwise the function that characterizes this distance is completely arbitrary.

The distance of point from the surface will be called the smallest distance from current point of surface. Let us determine the order of distance of point $Y(y_i)$ from surface

(27) f(x) = 0 ,

assuming that Y is infinitesimally close to surface. Then we may find infinitesimally close points $X(y_i + dy_i)$, that belongs to the surface. Inserting such point X coordinates in the equation of surface and finding expansion with respect to growing orders of dy_i, we receive

(28) $f(y) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dy_i + ... = 0$,

where values of partial derivatives shall be computed by placing $x_i = y_i$, and the dropped members are of order two and higher with respect to dy_i. Assuming that point Y is not infinitesimally close to singular point of the surface, where all $\frac{\partial f}{\partial x_i}$ vanish, at least one of these quantities has finite value in point Y. Consequently, the order of all quantities dy_i can't be higher than order of f(y). In turn, one can find the value of dy_i, the lower order of which is equal to the order of f (y) and satisfying equation (28): e.g., all dy_i may be set to zero, except one, which coefficients are different from zero. The remaining dy_i are exactly of order of $f(y_i)$. Followingly, the distance of point Y to surface has the same order than quantity $f(y_i)$.

Let us consider curve L with current coordinates given as functions of parameter t

(29) $x_i = x_i(t)$,

and its points X_0 and X_1 that have parametric values t_0 and t_0+dt in correspondence, where dt is infinitisimally small quantity of first order. If all $x_i'(t_0)$ are not equal to zero, the distance between points X_0 and $X_1$is quantity of first order too. Assuming that point X_0 belongs to surface (27) , let us determine order of distance of X_1 to this surface. It may be done by replacing quantities x_i in the left side of equation of surface with their expressions (29) and determining the order of values of function F(t) of t received in this way, when $t = t_0$+dt. Because X_0 belongs to surface,

$F(t_0) = 0$

and by expansion $F(t_0 + dt)$ with respect to growing orders of dt, the order p of this quantity should be equal to smallest number k that

$F^{(k)}(t_0) \neq 0$.

Comparing characteristic of touching of curve and surface, denoting by X_0 the point that is common to curve L and surface V and isn't singular neither for one nor other, and with point of curve X_1 with distance from X_0 as first order infinitisimally small quantity, we see:

if curve L at point X_0 has p-1 st order touching with surface V, the order of distance of point X_1 from surface V is of order p;

reversely, if distance of point X_1 from surface V is of order p, curve L touches at point X_0 with order p-1.

Further, denoting by X_0point that is common to two curves L and L' and isn't singular neither to one nor other, and with poin X_1which distance to $X_0$is infinitisimally small of first order:

if curves L and L' touch each other at the point X_0, distance of each curve to point X_1 is infinitisimally small quantity of order p;

reversely, if point $X_1$of one curve has distance to other curve is infinitisimally small quantity of order p, both curves at point X_0 has touching of order p-1.

The first property is direct consequence to last definition of order of touching of two curves; the consequence of second property is first characteristic of order of touching of L and L'.

In analogous way, comparing orders of infinitesimally small distances, we could characterize order of touching of two any varieties in their common point too.

The concept of an infinitely small distance, as can be seen, makes it possible to characterize the order of touching in a purely geometric way without using in the task, seemingly unfamiliar analytic elements $---$ the parameters and derivatives of the function with respect to them. Besides, judgments and formulations become easier. For example, if we define ds by setting to zero homogenous form of ds and dx_i, in each point isotropic directions arise, i.e., ds can become zero also in case, if all dx_i are not equal to zero. Using such metric, formulations above shall be true with restrictions caused by existence of isotropic directions. To characterize the touch and in these exceptional cases, one way or another it may be necessary to use any parameter. Secondly, the metric of points assigns to the points an exceptional role among other geometric objects; excluding parametric images, we somehow give geometrical objects globally. Both of these circumstances may hide various analogies.

Using parametric images and not defining any metric, we are not dealing with any isotropic directions in the case of their absence. A curve is characterized as a geometrical place of its points, which should allow us, mutatis mutandis, to attribute its properties to families of surfaces of one parameter. For these reasons, when considering a touch, the basis is the consept of parametric image of a curve, and not the concept of distance.

7. It should be noted that all considered properties are invariant with respect to any continuous unique and sufficiently many times differentiable transformation of basic variety T, that is characterized by relations

(30) $\tilde{x_i} = \tilde{x_i}\ (x_i)$

between coordinates of some point X(x_i) and modified point $\tilde{X}\ (\tilde{x_i})$. Really, giving x_i as unique funcion of one parameter, $\tilde{x_i}$ too should be unique function of one parameter and reversely - transformation (30) transforms curves into curves and surfaces into surfaces. Quantities $x_i(t)$ and their derivatives up to order p, if $t = t_0$, are determined by values of quantities $\tilde{x_i}$ and their respective derivatives at the same value of parameter, and reversely. If for two curves some initial members of expansion are equal, and reversely, then for modified curves corresponding quantities will be equal; followingly, the order of touching of two curves and similarly, of any two varieties is conserved.

Touching of order p is transitive property: if curves L and L' has p-touching with some third curve L'' in this point. curves L and L' have in this point at least p order touching. Because of this we may use concept of element of touching to characterize order of touching. We define this by saying that two curves that have order p touching at point X have common order p element of touching with support X. Transformations (3) convert elements of touching of each order into elements of touching of the same order. According Erlangen program of Klein[2] we may say that related properties of orders of touching create geometry of elements of touching. In this geometry many theorems are valid, that in classical Euclidean differential geometry is seemingly metrical character. We are to mention two of them.

If two curves in some m dimensional variety Q at point X, that is not singular, p-fold touch m-1 surfaces, so that their intersection variety doesn't touch variety Q at point X, both curves has common element of touching of order p.

The theorem formulated in this way is trivial consequence of definition of p-fold touching of two curves, because hypotheses create special case of condition of this definition. Specifying for three dimensional Euclidean space, we get a familiar theorem that two curves of surface that has at point X common osculating plane has the same curvature there too.

For some surface V' to have p+1-fold touching with all curves of given surface V that pass through point X_0, and that has in this point common elements of order p, surface V' must obey to p+n conditions; in other words: in any family of surfaces of p+n parameters in general case may be found one or several (of finite amount) surfaces with the required property. Besides, surface touches surface V.

For all curves L passing through point X_0 with common touching element of order p in this point, as we saw in point 3, may be chosen such parametric images

(31) $x_i = x_i$ (t) ,

that for parametric value t_0, not depending from the choice, x_i (t_0) are coordinates of point X_0, and derivatives of all x_i against t up to order p has the same values. If curve L stays in the given surface V with eqution

(32) f(x) = 0 ,

replacing in the equation (32) quantities x_i with functions (31), the received eqution and its derivatives should be satisfied with any value of t. Notably, if t = t_0, should hold

$$(33)\quad \left\{ \begin{array}{ll} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} & = 0 \\ \dots\dots\dots\dots & \dots \\ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{d^{p+1}x_i}{dt^{p+1}} & +F_{p+1} = 0 \end{array} \right.$$

where quantity F_{p+1} is determined by values of partial derivatives of f against x_i up to order p+1, and derivatives of x_i against t up to order p.

Surface V', that depends from parameters a_j (j= 1,2,,...,N), we characterize with equation

(34) g(x, a) = 0 .

For surface V'to have with curve L touching of order p1, replacing in equation (34) x_i with functions (31), parametric value t = t_0 should satisfy received equation, as well as derivatives of order p+1:

$$
(35) \quad \begin{cases} \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \frac{dx_i}{dt} & = 0 \\ \dots\dots\dots\dots\dots & \dots \\ \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \frac{d^{p+1} x_i}{dt^{p+1}} & + F_{p+1} = 0 \end{cases} .
$$

Quantity G_{p+1} is to be determined similarly as quantity F_{p+1}, in place of function f taking function g.

Since for all curves L at point X_0 quantities x_i and their derivatives against t have the same values, equation (34) and first p of equations (35) each gives condition to quatities a_j. In general case, all these conditions will be independent, because each contains g partial solutions, that in previous didn't. The last equation from (35) should be satisfied for all these values of ${x_i}^{(p+1)}(t_0)$, that satisfy last equation of (35), so then coefficients of both these equations at point X_0 should be proportional, that gives n conditions:

$$
(36) \ \frac{\frac{dg}{dx_1}}{\frac{\partial f}{\partial x_1}} = \dots\dots = \frac{\frac{dg}{dx_n}}{\frac{\partial f}{\partial x_n}} = \frac{G_{p+1}}{F_{p+1}} \text{ at point } X_0 .
$$

If these conditions are satisfied, equivalence of first n relations show that first conditions in systems (33) and (35) are consequence of one to another, so that surfaces V and V' touches one another at point X_0. Since all curves L satisfy condtion (33), for determination of surface V' we have left n+p independent condtions in general case: (34), (35) exculing first, and (36).

Let us specify for three dimensional Euclid space (n=3), taking p=1, n+p=4. In this case, as surface V' we may take sphere that has with each curve L touching of second order. Intersection of sphere with osculating plane of curve L at point X_0 is osculationg circle in this point, where from Meusnier's theorem follows.

Next to elements of touching that characterize touching of curves, it may be considered these too that are characterized by touching of any two varieties, varieties of such elements, a.s.o. We will not deal with such issues especially. It should only be noted that the theory for this is closely related to the transformations of S.Lie of touchings, when such elements are transformed into each other, and the theory of differential equations. Each differential equation or system of them characterize any variety Q of elements of touching. From view of S.Lee to integrate differential equation isn't anything else than to form from elements of one variety another variety, that is subject to special rules[3].

8. The order of touching of surface V and curve L at point X_0 may be defined also in case, when X_0 is singular with algebraic character of singularity, i.e., if in this point either for equation of surface

f(x) = 0

disappear all partial derivatives from left side by deriving against x_i up to certain order, or disappear derivatives of all current coordinates

$x_i = x_i$ (t) i = 1, 2, ..., n

against t up to certain order, or both conditions are satisfied at the same time.

Let us assume that all points of surface are not singular, and that in neighborhood of point X_0 each point of curve has only one value of t. Equation

F(t) = f[x(t)] = 0

at $t = t_0$ should have multiple root, since derivative against t of function F(t) at point X_0 equals to zero because of singularity of point. Equating to zero first derivative of F(t), that don't disappear automatically, if $t = t_0$, we receive condition, for order of touching curve and surface to be at least one. Conditions for higher orders of touching we should receive by equating to zero further derivatives of F(t), if $t = t_0$.

Another type of singularity arise, if the same point X_0 give different values of parameter t_0, t_1,..., t_k. In this case curve has several branches, that pass through X_0; current coordinates of each branch we receive, considering values of parameter in neighborhood of each value t_h (h=1,2,...,k). Then we can determine for each separate branch order of touching with surface V, that passes through X_0. In general case theses orders may be different, so that here isn't possible to speak about order of touch of curve and surface - the kind of touch is characterized by collection of orders of touching of all branches.

In both considered cases, in point X_0 the number of points of intersection of surface and curve that coincide would be by one unit more than order of touch, respectively, order of touch of separate branches, and their sum. In order to simplify our argumentation, if not said otherwise, we are to exclude points of singularity.

§2. On families of surfaces of one parameter

1. Considering touching of curve L and surface, we firstly seek conditions for certain surface to pass through infinitisimally close points of curve. For this purpose, in the equation

(37) $f\,(x_1\,,\,x_2,\,...,\,x_n,\,a_1,\,,a_2,\,...,\,a_N) = 0$

of general surface A of family S, we took coordinates x_i of point X as functions of one parameter t, assigning values of argument t_i, t_2, ..., t_p, and required them to tend to one value t_0. Now we are to act differently: let us give quantities a_j as one parameter functions:

(38) $a_j = a_j$(t) j = 1, 2,..., N

In this way determined j characterize family R of surfaces of one parameter. Asigning values t_1, t_2, ..., t_p to parameter t

$$(39)\quad \left\{ \begin{array}{ll} G\,(x,t_1) & = 0 \\ G\,(x,t_2) & = 0 \\ .... & ... \\ G\,(x,t_p) & = 0 \end{array} \right. ,$$

where

(40) G(x, t) = f{x, a(t)] .

Let us force all surfaces characterized by equation (39) to tend to surface A_0, that is determined by $t = t_0$; each point that belong to all surfaces (39) in the limit should belong to surfaces

$$(41)\quad \left\{ \begin{array}{ll} G\,(x,t_0) & = 0 \\ G'\,(x,t_0) & = 0 \\ .... & ... \\ G^{(p-1)}\,(x,t_0) & = 0 \end{array} \right. .$$

In analogy with the notion of infinitisimally close points, also surfaces with differences of parameter a_j infinitisimally small we may call infinitisimally close surfaces. If relationship (41) holds, and besides

$$(42)\quad G^{(p)}\ (\mathrm{x},\ t_0) \neq 0\ ,$$

we may say that point X is contained exactly in p infinitisimally close surfaces of family R, that coincide with surface A_0.

Equations (38) to (41) by their structure correspond to equations (4) to (7) from previous paragraph, only roles of both series of quantities x_i and a_j are exchanged, i.e., characterizing numbers of points X and surfaces A. Performing the same replacing and exchanging also numbers n and N, that express number of x_i and number of a_j, we could rewrite all equations of the whole previous paragraph. The transformed would express properties that arise from correspondind properties of original equations exchanging roles of points X and surfaces A.

We have come to certain kind of principle of duality: general relations that connect infinitisimally close points and surfaces of family S have in correspondence dual relations where points and surfaces have exchanged roles, so that connect infinitisimally close surfaces and points. If n=N, this duality is complete - varieties of points of some dimension m should have variety of surfaces of the same number of dimensions. If by contrast n≠N, different ways how we can characterize variety of m points may cause it to have two different kinds of varieties of surfaces in corresponce. Really, we can determine Q by giving all x_i as functions of m independent parameters, or else by determining n-m independent relations between x_i. By replacing x_i with a_j , in the first case we are to get variety of surfaces with m parameters, but in second case - variety with N-n+m parameters. First representation may give trivial results, if m⩾N, likewise second too, if N-n+m=0; if N-n+m<0, second representation would not have any sense.

The mentioned principle is anything new at all. By integrating equations of Pfaff, e. g., not only points and surfaces, but any point variety in general is to be considered as equal formation ("ldzvērtgs veidojums")[4]. But while in mentioned and similar cases are considered usually some certain order elements of touch, we consider in all conditions within give problem possible orders of touch.

2. Let us transform some concepts that we encountered in the begining of previous paragraph. In order to simplify text, determining parameters a_j of surface A we will call coordinates; if coordinates of point X and surface A are connected with relation (37), we will say that point and surface incide or are incident. Incident points X of one surface dually correspond to surfaces that incide with one point X; curve L as geometric place of points has in correspondence family R of surfaces as geometric place of surfaces.

From consideration excluded singular points of surfaces and curves are giving: for points X, singular surfaces of families of incident surfaces that lose all $\frac{\partial f}{\partial a_j}$ and singular surfaces of family R that lose all $\frac{\partial a_j}{\partial t}$; the latter we will call stationary surfaces. Also here we will exclude from consideration at all both type singular surfaces.

Let us consider further some concepts that may be connected with family

R, and let us determine their dual transformations. For all surfaces (41), their common points we will call characteristic points of order p-1 of surface A_0. The variety created from these points we will call characteristic of order p-1 of surface A_0; in general case it has n-p dimensions, since it is determined by p equations between x_i; the number of dimensions will be greater, if one or several equations are caused by others. By constructing for each surface of family R characteristics of order p-1, their collection in general case will create variety of order n-p+1, that we will call envelope Q_{p-1} of order p-1 of family R.

Let us find out that Q_{p-1} touches with order at least p-1 each surface of family R in each its characteristic point of order p-1; this fact doesn't depend from number of dimensions of Q_{p-1}. To prove this, it sufices to find out that each curve L in the variety Q_{p-1} through some characteristic point of order p-1 of surface A_0 touches in this point surface A_0 with order at least p-1. Curves in variety Q_{p-1} which all points have the same value t_0 in correspondence belong to one surface A_0. It remains thus to consider only curves different points of which have different values of t_0 in correspondence. By denoting these changing values with t, we may treat current point X coordinates of curve L as functions of t. These functions, inserting $t = t_0$, for each value of t_0 satisfy equation (41) - we may say that they are obtained from the first of them that are derivated against t_0;it should be proved that, inserting $t = t_0$, p-1 first derivatives against t of first equation are satisfied too. Since we are to derivate both against t and t_0, for the sake of simplicity we will give to last quantity a new designation s. The proved property obtain following formulation, denoting partial derivatives with corresponding power of argument in the index:

if, inserting t=s, identically relationships

$$(42) \quad \left\{ \begin{array}{ll} G(t,s) & = 0 \\ G_s & = 0 \\ G_{s^2} & = 0 \\ \ldots & \ldots \\ G_{s^{p-1}} & = 0 \end{array} \right. ,$$

are satisfied, that the same value of t satisfies system

$$(43) \quad \left\{ \begin{array}{ll} G & = 0 \\ G_t & = 0 \\ \ldots & .. \\ G_{t^{p-1}} & = 0 \end{array} \right.$$

too.

Really, inserting t=s, and derivating in this way identically satisfied equation (42) , substracting the last, we see that

$$(44) \quad \left\{ \begin{array}{ll} G_s + G_t & = 0 \\ G_{s^2} + G_{\mathrm{st}} & = 0 \\ \ldots & .. \\ G_{s^{p-1}} + G_{s^{p-2}t} & = 0 \end{array} \right.$$

Since equations (42) show that t=s identically turns to zero first members of left side of this equation, identity

$$(45)\quad \left\{ \begin{array}{ll} G_t & =0 \\ G_{\mathrm{st}} & =0 \\ ... & .. \\ G_{s^{p-2}t} & =0 \end{array} \right.$$

holds too.

Derivating these with condition t=s identically satisfied equations, and comparing obtained equations with (45), we see that idetntically turn to zero also all partial derivatives of G up to order p-1, that arise by derivating two times against t, and other - against s, etc. At the end we see that t=s identically turn to zero not only left members of equations (43), but also all partial derivatives of function G up to order p-1 includingly.

The obtained result we may a little generalize: if condition t=s turn identically to zero function G(s,t) and also at one of its partial derivatives up to order p-1, it turns to zero all partial derivatives of G up to order p-1. Really, if p=2, this property follows from the first equation (44). If it holds up to order p=k-1, derivating all k-1 -order partial derivatives, we obtain k mutually independent equations that express that sum of two k-order partial derivatives vanish, if t=s. Since according hypethesis one of these derivatives vanishes, vanish also other ones too, causing mentioned property.

Rendering geometricall equivalence of systems (42) and (43), let us consider curve L with coordinates x_i(t) of current point X and family R of surfaces with coordinates a_j(s) of current surface A. Let us connect point X and surface A, that has t=s. If each point X belongs to p infinitesimally close surfaces of family R, that coincide with surface A, each surface A passes through p infinitesimally close points of curve L, that coincide with point X.

Followingly, earlier mentioned property of envelope is proved, that is consequence of just mentioned fact.

Dually transforming, family R and its general surface A gives curve L and its general point X. p-order characteristic points of surface A have in correspondence surfaces, that we may call p-order characteristic surfaces; they pass through p+1 infinitesimally close point of curve L. Finally, characteristics correspond to families of characteristic surfaces. Properties of these surfaces families we may be obtained by dual transformation properties of characteristics.

Now we may give also characteristic to singular surfaces A, that incide with point X_1 and with all $\frac{\partial f}{\partial a_j}$ vanishing: for these surfaces, not depending from choice of a_j(s), both conditions (42) with p=2 are satisfied: first says nothing else than incidence of A and X_1; in second all derivatives of coefficients a_j vanish. Point X_1 thus is in all first order characteristics of surface A. Surface A in this point, if it is not singular, because of this, touch first order envelope of each one parameter family R of surfaces, that contain X. Besides, from G_s=0 follows G_t=0; thus each curve L that consists only from considered points X_1in each of its points touch corresponding surface A. Geometric place of all points X_1 we may call common envelope U of family T, since this surface touch in points X_1corresponding surfaces A. This property follows from equivalence of systems (42) and (43), where p=2. As we saw, for each point X_1 condition (42) is satisfied. Taking family of one parameter points X_1, i. e., curve of surface

U, (43) shows thatthis curve touches surface A. Thus, for singular points X of separate surfaces correspong with dual transformation singular surface A that in considered point X_1 touches common envelope of family T.

3. If family R of surfaces A is given arbitrary, maximal order of its characteristic points in general case is n-1. In dual way transforming, let us order conclusions of previous paragraph points 2 to 4. Rewriting system (41) for case p=n and replacing t_0 with s, obtained system

$$(46)\quad \{ \begin{array}{ll} G(x,s) & =0 \\ G'(x,s) & =0 \\ ... & ... \\ G^{(n-1)}(x,a) & =0 \end{array}$$

have equations with n unknowns x_i. If identically holds

$$(47)\quad \frac{D(G,G,\ ...,\ \text{SuperscriptBox[G, (n - 1)]})}{D(x_1,x_2,...x_n)}=0$$

for any values of x_i, system (46) may be solved with respect to x_i, obtaining them as functions of s. If equations (46) are not all linear with respect to all x_i, we get several systems of solutions. For each system corresponding point X will be called shorter characteristic point (not n-1 -order characteristic point) of surface; it is determined in general case by n infinitesimally close surfaces of family R. If values x_i determined by system (46) satisfy also relations

$$(48)\quad \begin{array}{c} G^{(h)}(x,s)=0 \\ G^{(n+q)}(x,s)\neq 0 \end{array} \qquad \text{h = n, n+1,...,n+q-1}$$

point X is staying in exactly in n+q infinitisemally close surfaces A of family R. It will be characterized saying that point X is q-order supercharacteristic point.

Inserting given by system (46) values of x_i in conditions (48), each of them will give on solution of system against s. For arbitrary given family R, roots of first equation (48) will not satisfy second, that's why in separate surfaces first order supercharacteristic points will be possible; higher order points of this kind in general case will be absent.

If on contrary all surfaces of family R pass through one or several (finite in number) points X, they will be considered as arbirary high order supercharacteristic points. Last example of mentioned 4. part of previous paragraph show that there exist one parameter family of surfaces with arbitrary high order supercharacteristic points.

Finaly, let us formulate right there mentioned problem dual transformation: if family S of surfaces A is given, is there possible to unite them in one parameter families, where each surface has q-order supercharacteristic point, so that q$\geqslant$1 and finite.

For a while postponing this question for more detailed examination, let us note that for both problems:

to find curve L with maximal order supercharacteristic with one given family S surface in each of its point, and

to find in family S contained one parameter family R of surfaces, that have maximal order characteristic points

has the same solutions, if they exist at all. They are given by curves L and their osculation surfaces A of family R. Really, as existence of already several times utilized systems (42) and (43) show: touching order of of some curve L with osculating surface A in its current point X is equal with poin X, as characteristic point of surfaces, order. If one of these numbers obtain in general possible maximal value, also second becomes maximal.

4. Elements of basic variety T in the beginning of this work, that we due to conveniency called points, we characterized only by possiblity to assign them coordinates x_i. In similary way we characterized also surfaces of family S with coordinates a_j and incidence condition

(49) f(x, a) = 0

of point X and surface A.

In examples we mentioned afine and Euclid space, assigning to words "point" and "surface" their usual meaning. With same virtue we could with words "point" and "surface" denote geometric objects of any two kinds, that may be characterized by coordinates, and with equations between coordinates expressing some geometric feature. So, e. g., for "point" we could take line of Euclidean space, for "surface" - sphere, and with "equation of surface" (49) express that line intersects sphere under some defined angle. All properties already found would hold, expressing them in suitable way.

In this way considered elements of theory of touching open way for essentially identical but in form very different geometric researches.

Mentioned changing of names of objects we could have performed here too, exchanging words for objects, that we called points and surfaces - with such changing we could have achieved the same as with utility of dual transformation - both basic objects would be replaced in roles.

§3. Determination of the order of solution of equations

If eqution with one unknow a

(50) f(a) = 0

has r-order root $a = a_0$, as it is known, holds:

(51) $f\,(a_0\,) = f'\,(a_0\,) = \;f"\,(a_0\,) = ...= f^{(r-1)}\,(a_0\,) = 0$,
$f^{(r)}\,(a_0\,) \neq 0$.

Expressing a as invertible unique funcition of some other quantity t

(52) a= ϕ(t) ,

so that $a_0 = \phi\,(t_0\,)$ and $\phi'\,(t_0\,) \neq 0$,

both system, composed from (50) and (52) , if (51) holds, $t = t_0$, $a = \phi\,(t_0\,)$ has r-order system of solutions. Considering a as coordinate of point of one dimensional variety, using language of previous paragraph we may say that surface (50) and curve (52) has r common infinitesimally close points. They coincide with point that has coordinates $a_0 = \phi\,(t_0)$.

Let us seek analogous criterion that allows to determine order of solution to system of equations that determine quantities a_j

(53) $f_k\;(a_1,\;a_2,\;...,\;a_N) = 0$ k = 1, 2,..., N,

with solution $a_j = a_{j0}$.

let us assume that for sufficiently small $|\; a_j$- $a_{j0}|$ following conditions ar satisfied:

a) $a_{\rm j0}$ form unique system of solutions of (53) (this condition holds always when total number of solutions of (53) is finite);

b) partial equations of functions $f_1,\ f_2,\ ....,\ f_{N-1}$ with respect to all a_j form matrix of rank N-1. If this condition is holds, if necessary renumbering quantities a_j, we may achieve that

(54) $(a_j) = \frac{D(f_1,f_2...,f_{N-1})}{D(a_1,a_2...,a_{N-1})} \neq 0$.

In order to characterize order of solution $a_j = a_{\rm j0}$, let us express all a_j as unique functions of one parameter t

(55) $a_j = \phi_j$(t) ,

and besides for some certain value t_0 of t

(56) $a_{\rm j0} = 0$ for all j

and

(57) $a'_j\,(t_0\,) \neq 0$ for at least one j.

Inserting values (55) of a_j in equations of system (53), we obtain equations that all have roots $t = t_0$. If it is possible to find such functions (55) that satisfy (56) and (57) in a way that order of root $t = t_0$ for each of equations is at least r, but it isn't possible to find such function that each root $t = t_0$of eqution (53) becomes greater than r, we will say that a_j is r-fold (of order r) solution of system (53).

Considering a_j as coordinates of point A of some N dimensional variety, we may characterize higher solution orders in a more geometric way: if it is possible to find curve for which point A_0 with coordinates $a_{\rm j0}$ is not singular and for which in this point coincide at least r infinitesimally close points of intersection with each of surfaces (53), and besides at least for one of surfaces this number is exactly r, but is not possible to find curve for which this condition would be satisfied for some r' that is greater than r, $a_{\rm j0}$ is solution of system (53) of order r.

Thanks to condition (54) for curve that gives maximally possible value of r, we may take first N-1 surfaces from (53), that we may call surfaces V, and call their curve of intersection L. Really, (54) show that system of equations that would be formed from these surfaces may be solved against $a_1,\ a_2,\ ....,\ a_{N-1}$, obtaining them for sufficiently small $|a_j\,,\ a_{\rm j0}|$ as a_N unique functions that becomes $a_{\rm j0}$ (j = 1,2,...,N-1) , when $a_N = a_{\rm N0}$. Expressing $a_N$as arbitrary invertible unique function ϕ_N(t), for which conditions (56) and (57) are satisfied, we obtain system (55) with required features (56).

Obtained values of $a_j$identically satisfy all equations (53), except last, for which roots r_0 according hypthesis are isolated and which because of this can't be identically satisfied. This gives some equation with respect to t

(58) F(t) $= 0$

where (59) F(t) $= f_N\ (\phi_1\,,\ \phi_2,\ ...,\ \phi_N)$.

Equation (58) has root $t = t_0$ of order r, if holds

$$(60)\quad \{\quad \begin{array}{ll} F\,(t_0) = F'\,(t_0) = F"\,(t_0) = ... = F^{(r-1)}\,(t_0) & = 0 \\ F^{(r)}\,(t_0) & \neq 0 \end{array} \quad .$$

Let us find out that no other curve L' can intersect each of surfaces (54) more than r infinitesimally close points that coincide with point A_0. Let's assume the opposite: curve L' intersects each of surfaces (53) at least at r+1 infinitisemally close point that coincides with A_0 . Because of conditions (54) and (57) point A_0 is singular neighter for surfaces V, nor curve L' too. Curve L' thus at point A_0 touches with order r each surface V and because of that their intersection curve L. But then, as it was seen, may be chosen such parametric images of curves L and L' that in point A_0 , where $t = t_0$, their derivatives against parameter up to order r of all corresponding current coordinates are equal. Then also by relations (59) defined functioin F(t), if $t = t_0$, together with their derivatives up to order r have the same values, not depending, either we determine it with help of parametric image of curve L, or curve L'. That's why $F^{(r)}$ (t_0) for both curves will be with the same value, and for curves L and L', contrary to assumption, correspond the same number of infinitesimally close with point A_0 coinciding points of intersection with last surface (53).

Let us express now conditions (60) only with help of functions f_k and quatities $a_{\mathrm{j}0}$. For this purpose, firstly let us compute derivatives against t of some a_j functions $H\,(a_1, a_2..., a_N\,)$, if a_j are given with relations (55), where ϕ_N(t) is arbitrary and other ϕ determined with the help of first N-1 equations of system (53). Derivating identically existing equations (53), we obtain N-1 equations

$$\sum_{j=1}^{N} \frac{\partial f_k}{\partial a_j}\frac{da_j}{dt}= 0 \quad \text{k= 1,2,..., N-1}$$

Solving this homogenous linear equation systme with respect to all $\frac{da_j}{dt}$, we obtain

$$\left\{ \begin{array}{ll} \frac{da_1}{dt} = & \rho\frac{D(f_1,f_2...,f_{N-1})}{D(a_2,a_3...,a_{N-1})} \\ ... & ... \\ \frac{da_j}{dt} & (-1)^{j+1}\rho\frac{D(f_1,f_2...,f_{N-1})}{D(a_1,a_2...a_{j-1},a_{j+1}..,a_N)} \\ ... & ... \\ \frac{da_N}{dt} = & (-1)^{N+1}\rho\frac{D(f_1,f_2...,f_{N-1})}{D(a_1,a_2...,a_{N-1})} \end{array} \right.$$

Factor of proportionality ρ should be determined by last equations, because $\frac{da_N}{dt}=\frac{d\phi_N}{dt}$ is known. In case its necessary, changing parameters, we may achieve that $\rho = (-1)^{N-1}$.

Then

$$\frac{dH}{dt}= \sum_{j=1}^{N} \frac{\partial H}{\partial a_j}\frac{da_j}{dt}= \frac{D(f_1,f_2...,f_{N-1},H)}{D(a_1,a_2...,a_N)}$$

By denoting

$$(61) \quad \left\{ \begin{array}{ll} D_1 = & \frac{D(f_1,f_2...,f_{N-1},f_N)}{D(a_1,a_2...,a_N)} \\ ... & ... \\ D_{i+1} = & \frac{D(f_1,f_2...,f_{N-1},D_i)}{D(a_1,a_2...,a_N)} \\ ... & ... \end{array} \right.$$

these functions are equal to derivatives against t of functions F(t) determined by relations (59):

$$D_i= F^{(i)}\ (\mathrm{t})$$

So then:

if conditions of beginning of this paragraph, and for values of a_j, $a_j = a_{\mathrm{j}0}$ are satisfied both conditions (53) and also coditions

$$(62) \quad D_1 = D_2 =... = D_{r-1}=0, \quad D_r\neq 0\ ,$$

system of values of $a_{\mathrm{j}0}$ is r-fold solution of equations (53).

Conditions (62) characterize r-fold solution also in case A_0 is singular point of last surface (53), because opposite hypothesis wasn't used anywhere.

In these conditions function f_k only seemingly has different roles. Each condition determines that matrix, where first partial derivatives of all functions f_k and left sides of previous condition are contained, has rank less than N; each besides should work as new condition for coefficients and quantities $a_{\mathrm{j}0}$ of function f. If point A_0 is not singular for last surface (53), in place of D_{i+1} may be taken functional determinant that is calculated, replacing in expression D_1some of functions f_k with D_i, only satisfying that rank of matrix of remaining first partial derivatives of f_k should be N-1.

If at point X_0 the rank of matrix of first partial derivatives of all functions f_k is less than N-1, our criterion is of no use of course, because values of all quantities D_i at this point are equal to zero. Also in this case, using before used technique, we could find suitable criterion, what we are not going to do in order not to distract too far from main subject.

By defining order of touching with help of infinitesimally small distance, as we have seen, we obtained the same number as using derivatives. Due to this, the criterion (62), if the condition of its utility is satisfied, is equivalent to the criteria of multiplicity of solutions of systems using the concept of an infinitesimal distance. Since this way we can characterize the multiple roots of a system of algebraic equations[5], our criterion is also useful in the case of algebraic equations.

It would be interesting to find out whether it is possible to establish usefulnees of criterion (62) and its generalization for systems of algebraic systems in purely algebraic way without use of concept of continuity - most posibly the answer should be positive.

§4. On criterions of order of superosculation and superosculation at each point of curve.

1. Let's first look at the conditions so that the curve L with its osculating surface at some particular point has a superosculation of finite order. We are giving curve L with current point $X\,(x_i\,)$ with equations

(63) $x_i= x_i(\mathrm{t})$ i = 1,2,...., n

and surface A that is dependent from parameters a_j (j=1,2,...,N) with its equation

(64) $f_1(\mathrm{x,a}) = 0$.

We rewrite equation that expresses osculation of curve and surface, highlighting in them orders of derivatives of x_1

$$(65)\qquad \begin{gathered} f_1(x,a)=0 \\ f_2(x,x',a)=0 \\ \ldots.. \\ f_N\left(x,x',...,x^{(N-1)},a\right)=0 \end{gathered}$$

Each of functions f_k we obtain by derivating previous agaist t, while we consider x_i as functions defined by (65) and a_j as parameters. Simbolically we may express this as

(66) $f_{k+1} = \frac{\partial f_k}{\partial x}\frac{dx}{dt}$ k⩾1

Equations (65) in general case give one or several a_j as systems of t valued functions. Let us take one of them. Corresponding surface A to have with curve L at point X at least first order superosculation, in this point besides should hold

(67) $f_{N+1}\left(x, x', ..., x^{(N)}\right.$,a) =0

Since functions x_i, their derivatives and a_j identically satisfy relations (65), we may derivate them, that gives, taking (66) into accoount

$f_{k+1} + \sum_{j=1}^{N} \frac{\partial f_k}{\partial a_j}\frac{da_j}{dt}$=0 k=1,2,..., N

Consequently due to relations (65) and (67)

(68) $\sum_{j=1}^{N} \frac{\partial f_k}{\partial a_j}\frac{da_j}{dt}$=0 k=1,2,..., N

System of N equations (68) is linear and homogenous with respect to quantities $\frac{da_j}{dt}$. It may be satisfied in two ways:

either

(69) $\frac{da_j}{dt}= 0$ for all j

in this case we say that surface A is stationary;

or determinant of system (68) is zero, that with designations of (61) express

(70) $D_1= 0$.

Seeking conditions for to occur in point X exactly r-1-order superosculation, two cases should be considered: $D_1\neq 0$, and $D_1= 0$.

2. Let us consider first case when $D_1\neq 0$. Besides equations (65), if exactly r-1-order superosculation occurs, at point X also should hold

$$(71)\quad \begin{aligned} &f_{N+1}\left(x, x', ..., x^{(N)}, a\right) = 0 \\ &\qquad \qquad .... \\ &f_{N+r-1}\left(x, x', ..., x^{(N+r-2)}, a\right) = 0 \end{aligned}$$

(72) $f_{N+r}\left(x, x', ..., x^{(N+r-1)}, a\right) \neq 0$

Deriving i times with respect to parameter t identically satisfied equations (65), we obtain equations of kind

(73) $f_{k+i} + \sum_{j=1}^{N} \frac{\partial f_k}{\partial a_j}\frac{d^i a_j}{dt^i}$ $+F_{\mathrm{ki}}$=0 k=1,2,..., N

Functions F_{ki} are sums of monoms that contain as factors derivatives of a_j up to order i-1, and their multiplications. Since determinant of coefficients $\frac{d^i a_j}{dt^i}$ for N equations (73) is $D_1 \neq 0$, setting in order i=1,2,..., r , and taking into account (65) and (71), conclude that at point X

(74) $\frac{d^i a_j}{dt^i} = 0$ $\quad \begin{matrix} i = 1, 2, ..., N \\ j = 1, 2, ..., r-1 \end{matrix}$

and

(75) $\frac{d^i a_j}{dt^i}$ $\neq$0 for at least one j.

Reversely, if condition (65) is observed in each point of curve, and besides (74) and (75) at point X, relations (73) show that (71) and (72) hold. So that:

in order osculating surface to have with curve L at its point X exactly r-1-order superosculation, it is sufficiently that conditions (74) and (75) hold, and (70) is wrong.

If osculating surface A with curve L in each of its point has at least first order super osculation and throughout $D_1\neq 0$, (69) shows that a_j are constant. In such case curve L sits on one determined surface A, and order of superosculation is infinite.

3. Let us now consider case when values of a_j obey (70). If matrix

(76) $\| \frac{\partial f_k}{\partial a_j} \| \qquad \begin{matrix} k = 1, 2, ..., N-1 \\ j = 1, 2, ..., N \end{matrix}$

has rank N-1, as we are to assume further, order of superosculation is at least r, where r is number for which at point X hold (74) and (75). Really, equations (73) with k=N and i=1,2,...,N-1 give equations (71). Setting i=r, and k=1,2,...,N, and observing obtained equations (71), we obtain relations

(77) $\sum_{j=1}^{N} \frac{\partial f_k}{\partial a_j} \frac{d^r a_j}{dt^r} = 0 \quad$ k=1,2,..., N-1

(78) $f_{N+r} + \sum_{j=1}^{N} \frac{\partial f_N}{\partial a_j} \frac{d^r a_j}{dt^r} = 0$

Since $D_1 = 0$ holds, and matrix (76) has rank N-1, from equation (77) follows

$\sum_{j=1}^{N} \frac{\partial f_N}{\partial a_j} \frac{d^r a_j}{dt^r} = 0$

because this equation is linear combination of (77), thus

(79) $f_{N+r} = 0$

If superosculations occurs at separate point X of curve L, condition that allows to determine its exact order, can't be expressed yet only with equation (65) and help of derivatives a_j, but equations (71) and (72) should be taken as help, or kind of similar equations. Thus, in this case we are to keep conditions (71) and (72).

4. More interesting than previous cases is case when osculating surface is not stationary and it has in each point of curve L r-1-order superosculation. In this case by equation (65) determined values of a_j identically satisfy relations (71), and (72) holds. Replacing in functions a_j argument t with s, left sides of equations (65), (71) and (72) are functions of f_1 and its partial derivatives against t up to order N+r-1. All these quantities, except last, identically vanish, if s=t. Then, as we have seen, in part two of paragraph 2, all partial derivatives of function f_1 against t, s or both arguments up to order N+r-2 including are identically zero, s=t, but N+r-1-order partial derivatives all differ from zero. Especially derivatives of left sides of against s up to order r-1 vanish, but $\frac{\partial^r f_N}{\partial s^r} \neq 0$. In previous paragraph we saw that this condition together with assumption about rank of matrix (76) and

(80) $\frac{da_j}{ds} \neq 0$ for at least one j

characterize r-fold solution of system (65).

Reversely, if a_j is r-fold solution of system (65), conditions about matrix (76) are in force, and (69) doesn't hold for at least one j, it may be concluded that (71) and (72) hold. Really, from (65) follow that

$\sum_{j=1}^{N} \frac{\partial f_k}{\partial a_j} \frac{da_j}{ds} = 0 \quad$ k=1,2,..., N-1

and (70) expresses that

$\sum_{j=1}^{N} \frac{\partial f_k}{\partial a_j} \frac{da_j}{ds} - \frac{\partial f_N}{\partial s} = 0$

is consequence of previous equation, that in turn causes

(81) $\frac{\partial f_N}{\partial s} - f_{N+1} = 0.$

In order to express higher order superosculation, we thus may replace equation (81) and its partial derivatives against t with equation (70) and its derivatives, because, holding (65), first follows from second. If besides

(82) $D_2 = 0$,

first derivative of equation (70) follows from equations (70), thus derivatives of (70) we may replace with (82) and its derivatives, and os on. From

$D_{r-1}= 0$

we conclude that partial derivations of all D_i (i=r-2,r-3,...,1) of order r-i-1 vanish, thus giving

$f_{N+r-1} = 0$.

Finally,

$D_r \neq 0$

together with (80) show that D_i (i=r-2,r-3,...,1) partial equations against t of order r-i don't vanish, that gives

$f_{N+r} \neq 0$.

So then: if osculating surface is not stationary and its rank of matrix of values of parameters is N-1, to occur at general point X exactly r-1-order superosculation, is necessary and sufficiently that A is exactly r-order osculating surface, i.e., its values of parameters are r-fold solution of system (65). In special points, where surface A becomes stationary, or else represents more than r osculating surfaces, order of superosculation in general should increase.

In general case, if system (65) may have r-fold solutions, it is not expectable that corresponding surface A is not changing and corresponding curve L in general sits on some non changing surface A; with concrete examples of such cases we are to meet in next chapter.

5. If in turn system (65) don't have r-fold solutions, requirement for superosculation of order r-1 in each point of curve L may be equivalent with condition that curve L sits in one or several non changing surfaces A.

Let us consider simplest corresponding case, when equation of surface A is linear against parameter a_j , as it is in all classical figures of osculation: for line, circle, conic in plane, plane and sphere in space:

(83) $\sum_{j=1}^{N} a_j\phi_j(x) + \phi_{N+1}(\mathrm{x}) = 0$.

Setting

$a_j = \frac{b_j}{b_{N+1}}$

with homogeneous parameter b_h equation of surface A may be written

(84) $\sum_{j=1}^{N} b_h\phi_h(x) = 0$.

Requiring to occur at least first order superosculation, except equation (84), where current point coordinates x_i of curve L are replaced with corresponding t functions, should hold also equation that we obtain k times (k=1,2,..., N) derivating (84) against t. Obtained linear homogeneous N+1 equation systems with respect to determinants of b_j that are Vronsk determinants of functions ϕ_h [x(t)], should be equal to zero. If rank of this determinant for some interval of values of t is N, follow relation of kind

(85) $\sum_{j=1}^{N} c_h \, \phi_h[x(t)] = 0$,

where all constants c_h are not equal to zero; if rank is N-1, follow several relations of kind (85), etc. In any case, if necessary dividing values of t in intervals, for each interval should correspond at least one relation (85); corresponding arc of curve L sits in surface (84), where

$b_h = c_h$.

If functions ϕ and x are analytic functions of their arguments, each relation (84) holds for all interval of t values, where these functions are defined, if it holds for arbitrary small part of this interval. By contrast, requiring only that functions x_i have continuous derivatives against t up to finite although arbitrary great order p, it may happen that different arcs of curve L has in correspondence different surfaces (84). As example we may mention following curve: in three dimensional Euclid space we take two spheres with common real circle C, and we construct in each sphere arc that in the same point X_0 of order p touches circle C, besides in a way that point X, passing through X_0 from one arc into second, doesn't change direction of movement; both arcs together form curve L that have for current point X coordinates continuous derivatives up to order p against suitable parameter, e.g., length of arc, and which arcs are in two different not changing spheres.

The property of family (83) of surfaces A that each curve that has in each of its point at least first order superosculation with some of surface of family all sits on one or several surfaces of family, of course, is independent from choice of parameters and arguments used. Taking coordinates y_i and parameters c_j of other type, in place of equation (83) should stand equation

(86) F(y,u) = 0 .

So, for example, equation of plane in polar coordinates

ρ cos(ϕ-α)=p

is not linear with respect to parameter α.

Naturally question arises: what should be the equation (86) so that it can be brought to the form (83), and with the help of what transformation it would be achieved; finally it should be checked either the obtained equation (83) display the same family than (86). The fact that the latter does not always occur can be seen by a trivial example of a family of one parameter: if equation (86) has only one parameter c_1, solving it against c_1, we get

(87) c_1 = f(y) ,

that corresponds to form (83), but it display family of other type than (86): if former surfaces have characteristics, the latter never have any. Difference between families displayed by (86) and (87), as it is known, arise because in general case function f in the latter equation is not unique function of its arguments y_i.

In principle simply, but because of length of computations practically unenforceable way, criterion for reduction of equation (86) may be obtained in the following way: taking quantities x_i as constants, this conditions relations between all a_j, namely, one of quantities, e. g., a_N, is linear function of all other. Let us characterize this relation with system

(88) $\frac{\partial^2 a_N}{\partial a_j \partial a_h} = 0$ j, h = 1,2,...,N-1 ,

that contain $\frac{N(N-1)}{2}$ independent equations. If transformation

(89) $a_j = a_j$ (c) ,

exist, that together with suitable transit from quantities x_i to x_i transform equation (86) to equation (83), taking one from c_j, e. g., c_N as function of other, condition (88) may be replaced with condition where come in partial derivatives against c_j of both first orders of c_N, and partial equations against c_j of both first orders of quantities a_j. These conditions should characterize relation (86) between quantities c_j, i. e., taking in these equations quantities y_i as constants, derivating it two times against all c_j, and excluding quantities y_i in between obtained equations, we are to get system of conditions of equal kind. By identification corresponding coefficients in both obtained systems, we obtain partial differential equation of second order for determination of unknown functions (89) of systems. The conditions for the integrability of this system are the searched criteria of possibility of reduction.

Using Pfaff form theory created by E. Cartan, J. Dubourdieu[6] has determined criteria of reduction for cases n=N=2. With the same question, or, more correctly, problem: equation

y'' = f(x, y, y')

to transform with point transformation into

y'' = 0

was engaged S. Lie[7]. Trying to find conditions for integrability, he has showed that reduction, if possible at all, require intergration of one linear simple differential equation if third order.

To find out either given family of surfaces belong to considered type, or not, practically most useful way seems to be consideration of system (65) for determination of osculating surfaces: if so, not depending from character of curve L, only one osculating surface is determined, though it would have in correspondence several or even infinitely mane systems of values of parameters (as, e. g., for line in plane, using polar coordinates), equation of family would be reducible to form (83).

6. We have to consider only two more cases, when values of a_j given by system (65) assign to matrix (76) rank less than N-1. In this case, corresponding osculating surface has in correspondence system of at least two-fold solutions of system (65). The touching order not always will be greater than N-1; if though superosculation happens, its order in general case will be more than one unit less than order of solution of (65). Thus case is analogous to curve intersection with line, if latter passes through multiple point; if though point of intersection becomes multiple, not always lines will touch curves. To determine exact order of eventual superosculation here, once more basis equation (71) should be used.

7. Transforming dually results of this paragraph, we are to get corresponding results for one parameter family of surfaces, and characteristic points of X. If corresponding to matrix (76) matrix has rank N-1, and characteristic point is stationary or multiple, it will be point of supercharacteristic. Notably, if condition about matrix is satisfied, and if coordinates of X are r-fold solution of its determining system, X will be r-1-fold supercharacteristic point. Finally, in part 5 of this paragraph considered families of surfaces have in correspondence surfaces of points that equations are linear with respect to coordinates. Taking x_i as affine coordinates, the considered problem obtain following formulation: determine when N parameter family of surfaces may be transformed, using point

transformation, into family of hyperplanes. That gives one more new criterion for possibility of this transformation: option should exist to create topologicaly equal configurations of figures of projective geometry with respect to curves of intersections of surfaces.

§ 5. On envelopes of maximal order

1. Assuming that the system (65) that determinate osculating surfaces gives sufficiently large amount of such surfaces, in general case it should be possible to find curve that has in each point at least n-1 order superosculation with some non stationary osculating surface. Really, in accordance with our assumption, system (65) may be solved solved against a_j , obtaining them as functions of i, x_i , x_i', ... , $x^{(N-1)}{}_i$. Setting these values in equations

$D_1 = 0 \;\; D_2 = 0 \;\; .... \;\; D_{n-1} = 0$

we get differential equation system that doesn't contain variable t, and is homogenous with respect to dt. Freeing from choice of parameter of indeterminance, we may take one from x_i as arbitrary function of t. Then system of n-1 equations arise that connect values of n-1 unknowns and their derivatives up to order N-1. In general case, if this system is not contradicting, and if it can be solved with respect to N-1 - order derivatives of all unknowns, curve L characterized by its derivatives will be dependant from (n-1)(N-1) arbitrary constants, for example, from initial values of all unknowns and their derivatives up to order N-2 for some parametric value t = t_0. In special cases of course may occur that it is possible to exclude all N-1-orders, eventually also further all unknown derivatives (???). Number of constants determining curve L will then decrease, or even such constants will be absent.

Curves L that are given by maximal compatible number of just considered equations we will call maximal order envelopes of family S of surfaces. As we just saw, it had in general case order of touching N+n-2. Thereby, its current point X is point of supercharacteristic of order n-1 of family of osculating surfaces. Configuration of curve L and family R could be obtained also considering quantities x_i in equations of family S as constants, a_j as functions of t, and derivating it n-1 times against t. If we want that determined x_i by obtained system are not constant and correspond to n-1-order supercharacteristic point, similarly as in previous paragraph, we are to find out that x_i should be N-fold solutions of this system of equations. Expressing this property, we obtain N-1 equations more. Excluding quantities x_i from all obtained equations, we obtain N-1 equations that contain a_j and its derivatives up to order n-1. They, similarly as higher considered differential equations for quantities x_i, allow to determine, if they are compatible, maximal order envelope L - values a_j just determine families R, and curves L are n-1 order envelopes of these families. Current point X of curve L represent N coinciding characteristic points of corresponding family R.

If system (65) may have several solutions, but their maximal possible order p is less than n-1, equations of number p that characterize realization of this maximal order, gives the same number of differential equations for quantities x_i. In this case maximal order envelope depends not only from arbitrary constants, but also from arbitrary functions.

Simple example, where besides once with existing "superelement" and second time without, is given by curve L of three dimensional Euclid space, and its family of spheres S, so that general surface depends from 4 parameters. Here, n=3, N=4. Since equation of sphere may be made linear with respect to a_j, none of curves L have superosculating non-stationar spheres. Three infinitesimally close spheres A determine two characteristic points, general sphere of one parameter family R has second order touching with geometric place of each characteristic point. Choosing family R in a way that both characteristic points coincide at one supercharacteristic point, sphere has third order touching whith its place in L. Maximal in general possible multiplicity 4 of characteristic point is not reachable, but family R and maximal order envelope L depends from two arbitrary functions.

With other examples we will meet in second chapter.

2. The problem of maximal order envelope and in general of superosculation has close contact with singular solutions of certain type of differential equations. In this work considered cases, when changing quantities are one parameter functions, corresponding equations should be those that determine following functions, namely, simple (n=2) and certain special Mong's (n>2) differential equations.

Let us consider first case, when n=2. In this case, family S is N parameter family of curves A; each curve A is characterized by constant values of a_j, thus quantities x_i, expressing them as functions of t, satisfying equations (65) and (67). Excluding from them quantities a_j, we obtain N-order differential equation for quantities x_1, x_2 :

(90) g(x, x', ..., $x^{(N)}$) =0.

First equation (65), with constant a_j, gives general solution of this equation. For all curves L, that have non-stationar superosculating curves A, also current coordinates satisfy equations (65) and (67), so then also (90). Since to these curves corresponding quantities e_j are not constant, they doesn't go into general solution of (90). Thus curves L display singular solutions of equation (90). As we have seen, in general case curves L form N-1 parameter family of curves, that in each their point X touch with order N some curve A.

Curves L may be determined also directly from equation (90). Since singular points were from our consideration excluded, when necessary changing coordinates, we may achieve that curve L under consideration has x_1'(t) $\neq 0$; x_1 may be taken as parameter t, x_2 denoted x. Then equation (67) is linear against $x^{(N)}$. Systems (65), where unknowns are a_j, and (65) together with (67), where unknowns are a_j and $x^{(N)}$ have their corresponding solutions of the same multiplicity. But then also each value $x^{(N)}$, that belong to multiple solution of systems formend from (65) and (67), should be multiple root of equation (90). Opposite feature should not always be true: $x^{(N)}$ value , that is (90) multiple root, may have in correspondence several different systems of values of a_j. Only fact is safe that for all elements of touching of order N of curve L corresponding value $x^{(N)}$ is multiple root of equation (90). Expressing with remaining members of equation (90) condition that characterize existence of multiple root $x^{(N)}$, we receive N-1 order differential equation that is satisfied by all elements of order N-1 touching of curve L. Integrating this equation, its general integral display N-1 parametric family of curves. This integral together with eventual singular integrals besides curves L can display geometric places for points that are singular for curves A, where for different curves A are the same value of $x^{(N)}$, etc. In each separate case should be checked, if and which obtained part of family consist from curves L. If equation (90) is given arbitrary, and not obtained with the help of exclusion of constants, as it is known, in general case singular integrals and thus also curves are absent[8].

In case n>2, from equations (65) and (67) may be excluded, similarly as before, all quantities a_j, obtaining equation of type (90) that now contain n>2 quantities x_i and their derivatives against t, i.e., Monge's differential equation. This equation doesn't have anymore form of corresponding type of equation, as we had in case n=2, because general Monge's differential equation doesn't have first integrals. Obtained equation however has N first integrals, that are given by expressions a_j that are computer from eqution (65). Besides, this first integral has special character, because from them all derivatives of x_i may be excluded, getting first equation (65).

Here too, maximal order envelopes L, if they exist, may be obtained by help of equation (90), utilizing similar considerations as in case n=2. If curve L has with non-stationary surface A r-1 order superosculation, N-1 order elements of touching of curve turn system of values of a_j determined by system of equations (65) into r-fold solution. Equation (67) is linear with respect to all ${x_i}^{(N)}$; giving also all ${x_i}^{(N)}$ except one, e.g., ${x_n}^{(N)}$, corresponding systems of values of $a_j$and ${x_n}^{(N)}$ formed by equations (65) and (67), should form r-fold solutions. Considering ${x_n}^{(N)}$ in equation (90) as unknown, this equation too should have at least r-fold root. Similarly each other value of ${x_n}^{(N)}$ should be at least p-fold root of equation (90). In the considered element of touching it should be

(91) $\frac{\partial^k g}{\partial {x_i}^k} = 0 \qquad \begin{matrix} k = 1, 2, ..., r \\ i = 1, 2, ..., n \end{matrix}$

Conditions (91) in general shouldn't be independent, because they are consequence of

(92) $D_1 = 0 \; D_2 = 0 \; \ldots \; D_r = 0$

that characterized also r-1 order superosculation. In any case however, should be possible to exclude all ${x_i}^{(N)}$ between equations (90) and (91), obtaining maximum r independent condition that connect x_i and their values of derivatives up to order N-1, and are conclusions of condition of (92). Writing condition (91) for maximal value of r, at which (90) and (91) form soluble system and it integrating, we obtain family of curves that in any case should contain at least maximal order envelopes, but can contain different other curves too, similarly as in case n=2. Similarly as before, here too, varieties that at each point (91) hold with r⩾1 correspond to singular integrals of equation (90).

Passing by, we have obtained interesting result: each N parameter family of surfaces may be characterized with Monge's (respectively simple ∂, if n=2) differential equation (90), that is satisfied by current coordinates and their derivatives of arbitrary curve of each family of surfaces.

The question arises: what Monge's equations (90) characterize family of surfaces. Without doing computations, let us make some possible ways that could give answer. Let us first observe that equation (90) should be homogenous with respect to all ${x_i}^{(N)}$, because it is equal to of the sort to equation (67) , where a_j are replaced with own values. Solving equation (90) with respect to one of ${x_i}^{(N)}$, we are to obtain equation that is linear and homogenous with respect to all ${x_i}^{(N)}$ - if such equation can't be obtained, (90) is not of type considered. Further should be said that obtained equation , where all members are brought to left side, has there N integral factors, that fact makes left side exact differential of independent functions. ??? Finally should be said that from obtained N first integrals

(93) g_k (x, x', ... , $x^{(N-1)}$) = const. k=1,2,..,N

all derivatives of x_i may be excluded.

Finally, we could replace equation (90) with corresponding system pf Pfaff equations, express that from this corresponding integrals from first integrals (93) might be obtainable.

§6. The simplification of computations in case of fundamental group

1. Let us connect with the basic variety T some continuous S. Lie group of transformations of r parameters that be elementary transitive, i.e., allows convert one in other two arbitrary given tangent elements that may be characterized by at most r numbers. To examine general/generic cases, in place of unchangeable coordinate system it is convenient to take some changing coordinate system that is connected with the figure to be examined, namely, Cartan's movable reference system ("repere mobile")[9]. As such reference system could be taken any basic figure of variety with the following properties:

a) it is determined by r parameters;

b) the group is simply transitive with respect to this figure;

c) each figure F is left unchanged only by identical transformations.

In order to examine some point of variety, for each of its point X uniquely is attached some reference system F. We characterize, with the help of F, infinitesimal group transformation that transforms F into to point X infinitesimally close point X_i attached figure F_1. Such obtained numbers, that Cartan called relative components of motion, give differential forms and differential invariants of the group under consideration. This all allow rather simply to create differential geometry corresponding to this group.

Example: In the Euclidean space of three dimensions, as figure F may be taken mutually orthogonal unit vectors with common origin. Attaching possibly closely this figure to current point of curve L, and determining relative components of the motion of F, we get Frenet frame and formulas and, together with arc element and both lower differential invariants, - curvature and twist.

Considering some other geometric objects that are connected with the variety of points X under examination, we characterize them with relative coordinates with respect to reference system attached to the point X. Each relation between these relative coordinates, that is nothing other than invariants of the figure under examination, expresses some geometric property. This approach already before Cartan's general theory creation has been used in special cases, e.g., in differential geometry of Euclidean space and plane, it was widely used by Darboux[10] and Cesaro[11]. The notion of condition of immobility used by the last author, with proper generalization, is very useful in research of touching and infinitesimal close varieties intersections.

Let us consider criteria that characterize p-fold touching, if coordinate system is unchanged, and let us determine how they are to be modified in case system changes. It is naturally to assume that the family of surfaces S under examination are invariant with respect to fundamental group; we are going to do this in what follows.

In case of unchanging coordinates, we wrote incidence condition of point $X\,(x_i)$ and surface $A\,(a_j)$, and we derivated it p-1 times, taking x_i for coordinates of current point of curve L and a_j for constants, i.e., we used conditions:

(94) $\frac{da_j}{dt} = 0$ j=1,2,...,N.

Using movable reference system F, that is connected with some current point Y of the curve L', we do similarly p-fold derivation of incidence equation, assuming x_i to be relative coordinates of current point of curve L. Only now, in order to characterize that surface A is changing, the condition (94) doesn't suit, that should be replaced with the condition of immovability in the following form:

(95) $d_j/$ ds $= f_j(a, Y)$ j=1, 2,..., N.

Now we denote with a_j relative coordinates with respect to changing reference system. Now we do derivation not with respect to arbitrary parameter t, but w.r.t. length of arc of the group, that is detected by its differential - the lowest differential form connected with the curve touching element. Concluding, with writing $f_j(a, Y)$ we express that f_j depends not only from values a_j, but from differential invariants in the point Y of the curve L'.

That the immovable surface relative coordinates a_j should satisfy condition of type (95), show the following argument: a_j and $a_j +$ da$_j$ are relative coordinates of the same surface A that are connected with reference systems F and F_1 , determined by two curve L' infinitisimally close points Y and Y_1 . Infinitesimal transformation that changes F_1 to F is the infinitesimal transformation (Y) of the basic group. It changes surface A into some surface A_1 that has in the reference system F the same relative coordinates b_j that surface A in the system F_1. Thus,

$b_j = a_j$ +ds$_j$ j=1, 2, ..., N.

The inverse transformation (Y_1) of transformation (Y) that changes F into F_1 is characterized by the relative components of movement that are determined by values of differential invariants of curve L' in the point Y and its arc element ds. Characterizing numbers of transformation (Y) are opposite to characterizing numbers (Y_1); applying this to some surface A with coordinates a_j, we get surface with coordinates

$c_j = a_j$ $+\phi$ (a,Y) ds j=1, 2, ..., N ,

where symbol Y as argument has the same meaning as before. Because b_j and c_j characterize the same surface A_1, relation (95) follows with $f_j = \phi_j$.

Forcing curve L'to coincide with curve L, coordinates of the point X become constants and the incidence equation of this point and surface contain only relative coordinates of the surface. By derivation, we are to get relation that besides these coordinates should contain only curve L differential invariants that are contained in the relative coordinates, and derivation with respect to s of the latter. It is clear that equations obtained in this way should be incomparably dimplier than using immovable system of coordinates.

Parameters of osculating surface we are to get by solving system that is composed from incidence equation and its N-1 first derivatives with respect to s. Putting obtained values of a_j into further derivatives, we are to get conditions for superosculation. Equations (95), in place of former (94), characterize stationary surface. If left side partial derivatives with respect to a_j of first N-1 equations that determine osculating surface give a matrix of rank N-1, the p-fold superosculating surface that is not stationary has in correspondence p+1-fold solution of the system determined by it, and reversely. Multifoldness of solutions is characterized by conditions that interrelate differential invariants with their derivatives. Maximal number of these conditions, that determine solution of the system, is determined by envelope of upper order.

Computations could be still simplified starting not with condition of incidence that express that surface A goes through point X of curve L, but as starting point taking some surface A that already has p-fold touching in the point X. Such surface has less then N number of free arbitrary parameters, because of touching in between a_j are realized p+1 relations. Expressing that such surface A is unchanging in infinitesimally small transformation that F turns into F_1 we receive both conditions of immobility for remaining free parameters and one relations in between those that characterize p+1-fold touching. Derivating this relation, with the help of condition of immobility, we are to get conditions that the order of touching is p+2, p+3, a.s.o. Concrete examples of this approach we are to consider at the end of this paragraph.

2. If number of quantities a_j is N, excluding a_j from equation of incidence and its N first derivatives, we obtain Monge's equation, that characterize the family S of surfaces. As we see in the page 59, this equation comprise derivatives of coordinates up to order N - in case of fundamental group it should contain differential invariants up to order N.

If family S contains all surfaces that are congruent to one surface A, i.e., surfaces that we get from A by all group transformations, maximal value of N is r. It is directly r, if fundamental group doesn't have continuous subgroup that turns the group into itself; It is r-r', if exists such subgroup with r' parameters. The considered equation characterize curves that are contained in this surface A.

As example let us consider three dimensional Euclidean space. Fundamental group, that is group of transfer r=6. Differential invariants of curve are: curvature - of second order - and its derivatives, twist - of third order - and its derivatives. The equation characterizing curves of the surface should contain curvature and its derivatives - up to order k+1, twist and its derivatives - up to order k, where k=3-r.

For arbitrary surface k=3; for srew surface, including cylinder and surface of rotation, k=2; for rotational cylinder k=1; for sphere and plane k=0. These values of k correspond to curves that are contained in a definite surface and congruent to it surface. For the curves of definite type these values should of course be greater: for curve in rotational cone k=3, in rotational cylinder k=2, in sphere k=1.

In the principle the indicated technique gives general method to fix considered equation that may be called natural equation of curves of family S.

Droping case n=2 and very simplest cases, if n>2 (e.g., for Euclidean sphere in space or for plane), already before last excluding of a_j, we get rather complicate equation, so that in the end, in the best case, we could give the result not explicitely but in form of determinant. Thus it is expected that the equation itself we are after should be of complicate form.

Using different methods, that do not allow to go further than determinant too, respectively, exclusion of one quantity in between two equation, determination of this equation has been considered in several cases for n=3.

To find equation for curves in rotational cylinder seems firstly has tried H.Piccioli[12], using intervals of lines.

After receiving correct equations in the beginning, he gets equation of second order with respect to line direction cosines with coefficients of as sums of elements of different dimensions. To eliminate direction cosines equation should be derivated 6 times; received determinant of the system of equations equated to zero as if should determine the relationship to be found, though author himself indicates that the expression can be simplified, noticing that elements of colons are linearly dependent.. It is strange that Salkowski who was referee of the article, didn't find anything to oppose.

V. Hlavaty[13], determining surface in three dimensional Rieman's space with its fundamental tensors, using tensor calculus finds two equations, within which one auxiliary quantity is to be eliminated.

E. Cotton[14], according reference that doesn't mention method, gives correct k values (the work itself wasn't accessible).

In the case of Euclidean planes, E. Cesaro[15] characterize family of curves with equation between length of arc and radius ρ of curvature, that contains yet arbitrary parameters:

(96) $f(s, \rho, a_1, a_2, ..., a_{n-2}) = 0$

and claims that osculating curve of such family touches given curve n-fold. This statement is true only when within parameters directly or indirectly doesn't figure arbitrary auxiliary constant, that arise by change of the point from which arcs are counted: The equation

(97) $f(s + a_0, \rho, a_1, a_2, ..., a_{n-2}) = 0$

characterizes the same family that (96). it's not entirely clear either Cesaro acknowledges superosculation in each point of the given curve possible, or not. In speaking about touching order/foldness, he says (in translation of Kowalewski): Dies hindert jedoch nicht, dass eine solche Berührung wegen einer der Curve (M) innewohnender Eigentümlichkeit thatsächlich eintreten kann"-possibility of such superosculation seems to be acknowledged. Nevertheless, further we read: "... nur in besonderen Punkten von (M) kann es eintreten, dass die

Ordnung der Berührung die Zahl n überschreitet", that such possibility seems to exclude.

Derivating equation (96) n-1 times with respect to arc and excluding s and constants a_j, Cesaro obtains relation within first n curvature radii

(98) $F(\rho, \rho_1, ..., \rho_{n-1}) = 0$,

that characterize family of curves (96). Here once more is to be noted that such exclusion is possible only when condition about additive constant is satisfied. Secondly, as we saw, equation (98) characterize not only family (96), but also all curves L for which in each point superosculating family curve??. Writing (98) as differential equation, its general solution should give general curve of family, which are singular solutions of curve L.

3. Considering for one parameter family R of surfaces characteristic points of separate surfaces, we may arrange calculations dual to previous, i.e., to use changing reference system that is connected with general surface A of the family. In order not to find relative components of movement for this new reference system, we may use here the reference system that is connected with some auxiliary curve L' in the current point Y. With the use of such technique the examination of the geometric points simplifies, because obtained data may be easy compared with the results of research mentioned in the beginning of this chapter.

The principle of computations in all cases is the same: incidence condition is written and derivated, taking into consideration that now immovable should be considered point, therefore point X, not anymore coordinates of surface A should satisfy immobility condition. If reference system is directly connected with the surface A, the condition of incidence is expressible only with the coordinates of the point. By connection reference system with point Y, condition of incidence in general case should contain both point X and coordinates of the surface A.

Similarly how we previously found natural equations of curves that are contained in some surface of the family S, also here may be found natural equation in the family R of surfaces of one parameter that go through some point of variety T. This equation is to connect differential invariants of family R up to order n.

In the end are to be mentioned flaws of the method of movable reference system: technique by which distinct F is connected with current element of general variety doesn't work for special varieties where one or more differential invariants of lower order has values zero. Thus, e.g., Frenet formulas doesn't work for isotropic curves and curves in isotropic planes in Euclidean space. To consider such varieties, in another way determined reference systems F are to be taken, where also relative components of motion are expressible in other way. Sometimes one can help with casual calculation, expressing figure F and relative components sufficiently generally in order to apply results with suitable specification to all or at least several cases.

4. Let us consider some simple examples of computations. While we throughout considered one parameter families of surfaces or points, we are going to characterize relative motion components not with differentials, as Cartan is doing, but with derivatives with respect to arcs of corresponding group.

a) In the Euclidean three dimensional space, let us consider curves that Frenet formulas hold:

$$
(99)\qquad
\begin{aligned}
X' &= \vec{t} \\
\vec{t} &= \rho\vec{n} \\
\vec{n}' &= -\rho\vec{t} + \tau\vec{b} \\
\vec{b}' &= -\tau\vec{n}
\end{aligned}
\quad ,
$$

assuming that $\rho \neq 0$. The current point is denoted by X, $\vec{t}$, $\vec{n}$, $\vec{b}$ correspondingly unit vectors of tangent, main normal and binormal.

To find the sphere that osculates the curve, we take sphere that is tangent to curve. If the radius of the sphere is a, its center C is given by

C = X + a[cos ϕ $\vec{n}$ + sin ϕ $\vec{b}$],

where ϕ, the angle between vectors $\vec{n}$ and $\vec{XC}$, is the second parameter that determines the sphere. For sphere to be unchanging/fixed, a and C should be unchanging/fixed that gives the condition of immobility

a'= 0

ϕ'= - τ

and the condition to get second order touching/tangent

(100) 1 - a ρ cos ϕ = 0.

Mulitplying by radius of curvature R = $\frac{1}{\rho}$ and derivating, we get

(101) R' - a τ sin ϕ = 0.

Conditions (100) and (101) uniquely determine osculation sphere. Mulipying (101) by radius of twist $T = \frac{1}{\tau}$ and derivating, we get condition of superosculation

(T R')' + a τ cos ϕ = 0.

Excluding a and ϕ, we get natural equation of curves of the sphere

(102) (T R')' + $\frac{R}{T}$ = 0.

If the radius of the sphere is given, (100) determines osculating sphere, excluding ϕ from (100) and (101) we get natural equation for curves in the sphere with radius a:

(103) $(R'T)^2 + R^2 = a^2$.

The equation (102) doesn't have singular solutions, equation (103) gets them at

R = a,

that characterize envelope of maximal order for spheres with radius a.

b) In the same space, we consider family of spheres of one parameter. Taking as a reference system the Frenet frame of the geometric locus L of the center Y of sphere, we get equation for the sphere with radius a and current point X :

(104) $(X - Y)^2 = a^2$.

The condition of immobility of the point X is $\vec{X}$'=0. Derivating (104) and observing the condition, we get

- $\vec{t}$ (X-Y) = a a'

-ρ $\vec{n}$ (X-Y) + 1 = (a a')'

This eqution together with (104) determines two characteristic points X, that are symmetric with respect to osculating plane at point Y of curve L . Supercharacteristic point we get forcing these two points coincide, i.e., expressing that point X is in the osculating plane:

(105) $(\mathrm{a}\ a')^2 + R^2\ [(\mathrm{a}\ a')' - 1]^2 = a^2$.

Derivating again equation that determines the point X and excluding the components of the vector X - Y, we get the natural equation for spheres through one point

$(\mathrm{a\ a'}) + R^2\ [(\mathrm{a}\ a') - -1]^2 + \frac{T^2}{R^2} \left\{ RR'[(aa')' - 1] + R^2 (aa')" + aa' \right\}^2 = a^2.$

(115) is the singular solution of this equation.

It is easy to get condition (105) and its interpretation by derivating last three equations, taking as variables only X (on pages 28., 29. partial derivations with respect to t):

$$
(116) \qquad \begin{array}{l} \vec{X}'(X-Y) = 0 \\ \vec{X}'\vec{t} = 0 \\ \vec{X}'\vec{n} = 0 \end{array}
$$

Both last conditions are independent. For point X to be immovable, vector X-Y should be coplanar with $\vec{t}$ and $\vec{n}$, i.e., point X should be in the osculating plane. Modifying the family of spheres so that neighter a nor ρ changes, but curve L becomes plane, both last conditions in (106) show that point X becomes immovable/fixed[16].

Here we encounter both with example for singular case, that was excluded by us from general consideration, namely, when both first equations in (106) aren't independent, i.e., X-Y and $\vec{t}$ are colinear. In this case 'izejas' equations give

X = Y + a $\vec{t}$

-1 = a'

Thus, point X describes filarevolute of curve L. X isn't point of supercharacteristic, though it represents {'pārstāj ?'} two coincident characteristic points, because the last equation of (106) is satisfied in the case when L is line and X - immovable point of this line.

c) In the Euclidean plane we look for an osculating conic of curve L, taking as axes of references system tangent and normal of L in the current point X. Each point Y may be expressed in the form

Y = X + x $\vec{t}$ + y $\vec{n}$.

Taking $\vec{Y}$'= 0, with help of Frenet formulae (with $\tau = 0$) we get condition of immobility

x'= -1 + ρ y

y'= -ρ x

Conic that touches curve L in the point X we may give with equation

(107) a x^2 + 2 b x y + c y^2 + 2 y = 0 .

We us seek the condition of immobility of its coordinates a, b, c that are changing by moving the point X along L. We reach this by derivation of the equation of conic, observing the condition of immobility of points, and expressing that the equation

(a'- 2 b ρ)x^2 +2 (b' +a ρ - c ρ) x y + (c'+ 2 b ρ) y^2 -2 (a + ρ) x - 2 b y = 0

is the consequence of the equation of conic (107). It gives

a'= - a b + 2 b ρ
b'= - b^2 + (c-a) ρ
c'= - b c - 2 b ρ

and a condition that the touching is of second order

a + ρ = 0.

Observing in this way obtained upper values of conditions of immobility, we get the conditions of immobility of conic with the second order touching:

b'= - b^2 + c ρ + ρ^2
c'= - b c - 2 b ρ

and condition

- ρ'= 3 b ρ,

for order of touching to be three. Replacing b with the value obtained in this way, we get the touching condition of order four

9 c ρ^3 = -3 ρ ρ'' + 4 $(\rho')^2$ - 9 ρ^4,

that allows uniquely to find c and thus osculating conic. Expressing that c obeys too the condition of immobility, we get the normal/natural? equation of conic

9 ρ^2 ρ' -45 ρ ρ'ρ'' ? + 40 $(\rho')^3$ + 36 ρ^4 ρ' = 0.

Chapter II

On rotational cylinders osculating spacial curves

§1. Basic equations

Discussion of some special cases

1. In the Euclidean space of 3 dimensions rotational cylinder (further called simply cylinder) may be determined by giving some of its axis point C, arc unit vector $\vec{u}$ and radius a; further we assume a$\neq$0. If the current point of the cylinder is X, its equation is

$$(108) \quad (X-C)^2 - \left[\vec{u}(X-C)\right]^2 = a^2.$$

Equation (108) is the incidence equation of point X and cylinder. Taking X as some current point of the curve L and other elements of equation (108) as constant, derivating this eqution with respect to arc s of L, we could get first and further conditions of touching curve with cylinder. Because cylinder in the space is determined by 5 independent parameters, osculation cylinder should have touching of order four (i.e., 4-fold tangent).

Calculations become simplier if we take cylinder that already is touching curve, and we determine it with suitable coordinates.

If cylinder touches the curve in the point X, the normal to cylinder in this point is the normal of the curve too. Placing C in the point of its crossing point with the axis of cylinder, it holds

(109) C=X + a(cos ϕ $\vec{n}$ + sin ϕ $\vec{b}$),

where ϕ is the angle between vetors $\vec{n}$ and C-X. The unit vector of axis is perpendicular to the vector C-X that goes along the normal of cylinder, so that it can be expressed in the way

(110) $\vec{u}$=cos ψ $\vec{t}$ - sin ϕ sin ψ $\vec{n}$ + cos ϕ sin ψ $\vec{b}$,

where ψ is the angle between vectors $\vec{u}$ and $\vec{t}$.

The cylinder, determined in this way, to be immovable, a should be constant, vector $\vec{C}$' should be collinear with $\vec{u}$, and $\vec{u}$ should be unchanged. Deriving expressions (109) and (110) with respect to arc using Frenet formulae and expressing mentioned conditions, we get the conditions of immovability:

$$(111)\quad \left\{ \begin{array}{l} a' = 0 \\ \psi = \rho \sin\phi \\ \phi' \;=\; -\tau + \frac{\sin\psi\cos\psi}{a} \end{array} \right.$$

and the condition that the order of touching/tangent were equal to two:

(112) a ρ cos $\phi = \sin^2\psi$

Derivating this eqution with the help of conditions (111), we get

(113) a(ρ'cos ϕ + ρ τ sin ϕ) = 3 ρ sin ϕ sin ψ cos ψ

Derivating once more and taking into consideration expression (112), we get

(114) a[(ρ''-ρ τ^2)cos ϕ+(2ρ'τ+ρ τ')sin ϕ]=3 ρ^2 $\cos^2\psi$
-3 $\rho^2\sin^2\phi$ $\sin^2\psi$ +4(ρ'sin ϕ - ρ τ cos ϕ)sin ψ cos ψ.

Just the same equations we were to get derivating (108) and putting (109) and (110) in there. The values of a, ϕ and ψ, that satisfy first, both two first or all three equations, from (112) to (114), characterize the cylinder with second, third and forth order touching point in the point X with the line L correspondingly.

On the back of the page with hand:

(**)

The last techique is used by A.Tamerl[17], who seems to be the only up to now who more widely has researched rotational cylinder with higher order touching with spatial curve. He researches mainly some configurations of axes of cylinders, the geometric place of the point C, if the order of touching is three, and some special cases, determining the number of different osculating cylinders, but researches neighter their multifoldness, nor degenerated cylinders. As long the received results are already in Tamarl's paper, is pointed out in the end of this chapter.

Equations (112) and (113) can be solved with respect to a and ψ, giving

(115) a = $\frac{9\rho^3\sin^2\phi\cos\phi}{(\rho'\cos\phi+\rho\tau\sin\phi)^2+9\rho^4\sin^2\phi\cos\phi^2}$

(116) tg ψ = $\frac{3\rho^2\cos\phi\sin\phi}{\rho'\cos\phi+\rho\tau\sin\phi}$

These expressions uniquely determine a and tg ψ, if at least one of conditions

$$(117)\quad \left\{ \begin{array}{ll} \rho\sin\phi\cos\phi & \neq 0 \\ \rho'\cos\phi + \rho\tau\sin\phi & \neq 0 \end{array} \right.$$

holds. Putting values of a and tg ψ in the equation (114) we get equation of order six against t=tg ϕ:

(118) $\left[\rho^2\tau^2 t^4 - 3\rho^2\tau' t^3 + \left(5(\rho')^2 - 3\rho\rho\text{''}\right)t^2 - -2\rho\rho'\tau t + (\rho')^2\right]\left(t^2+1\right) - 9\rho^4 t^4 = 0$

In the general case, determined by (118) values of t are distinct, and their given ϕ values satisfy both conditions of (117); each of these values of t has one point C and two oposite directed vectors $\vec{u}$ in correspondence, that characterize only one determined cylinder. Thus, general spatial curve has six osculating cylinders in correspondence.

Without performing complete analysis of equation (118), we are to mark some properties of it.

One of the roots satisfy none of conditions (117) only in the case when one of the quantities, ρ, ρ', τ equals to zero. These special cases we are to examine further, for the moment asuming that all mentioned quantities differ from zero.

In the distinct point of the curve and, next to it, in sufficiently little piece of the curve, it is possible to chose real values of curvatures and their derivative values in such a way that all roots of equation (118) are distinct and besides with number of real roots 6,4,2 or 0, correspondingly. Thus, real curves are available with all osculating cylinders being imaginar.

The summ of angles ϕ_i, determined by equation (118), are equal to multiple of π. Realy,

$$\mathrm{tg}\,(\phi_1 + \phi_2 + .. + \phi_6) = \frac{S_1 - S_3 + S_5}{1 - S_2 + S_4 - S_6},$$

where by S_j is designated sum of all multiplications of tg ϕ_i with respect to j. Calculating right side of the expression with the help of coefficients of equation (118), counter is equal to zero and denomnator in general case differs from zero.

The coefficients of equation (118) depend from quantities ρ and τ, that may be arbitrary functions of s. Therefor, we may put two conditions on them to be satisfied for all points of curve L, e.g., we may require for equation (118) to be throughout with one threefold or two double roots, similarly to require for some osculating cylinder two quantities from a, ϕ , ψ to be given functions of s, i.e., to be constant, respectively, and so on. Using such conditions, if for obtained curves for each of osculating cylinders at least one of quantities a, ϕ, ψ, doesn't satisfy the condition of immobility, the curve can't be located on unchanging cylinder. Some of such curves osculating cylinders of which satisfy certain conditions we are going to consider in the following chapter.

2. Let us now consider cases when at least one of the quantities ρ, ρ', τ turns to zero in the distinct point X.

a) If $\rho = 0$, (112) gives $\psi = 0$: the axis of cylinder is parallel to tangent in point X. (113) and (114) turns into

$$a\ \rho\text{'}\cos\phi = 0$$

$$a\ [\rho\text{''}\cos\phi + 2\ \rho\text{'}\ \tau\sin\phi] = 0$$

If ρ'$\tau \neq 0$, the only solution is a = 0, i.e., all osculating cylinders have turned into tangent to L in point X. If $\tau = 0$, osculating cylinder becomes undefinite: its axis can be any parallel to tangent, if ρ'= ρ''=?0, or else arbitrary parallel to tangent in the plane of rectification, if one of quantities ρ', ρ'' differs from zero.

For remaining special cases too we may assume that $\rho \neq 0$.

b) If ρ'= 0, equation (113) is satisfied in two-fold way: either

$\sin\phi = 0$

or else

a $\tau = 3 \sin\psi \cos\psi$

In the first case we may take cos $\phi = +1$, (112) gives

a $\rho = \sin^2\psi$

and this value putting in equation (114) gives

(119)$\frac{\rho"}{\rho}$ $\sin^2\psi = (3\ \rho \cos\psi$ - $\tau \sin\psi)(\rho\cos\psi$ - $\tau\sin\psi)$

As follows from (119), this value of ϕ has in correspondence two, in general case distinct, osculating cylinders. If ρ ''= 0 and $\tau \neq 0$, they surely are distinct, because they are characterized by

(120) tg ψ= $\frac{\rho}{\tau}$ and tg ψ =3$\frac{\rho}{\tau}$

The second posibility to satisfy (113) gives four, in general case distinct, cylinders, that have roots of (118) with $t_i \neq 0$ in correspondence, and for which formulae (115) and (116) are suitable.

c) (with hand added: if $\rho \neq 0$, ρ'=ρ''=0)

If ρ ρ'$\neq$0, τ = 0, τ'$\neq$ 0, equations, (112) to (114), have solution

$\phi = \frac{x}{2}$, $\psi = 0$, a τ'= 3 ρ ,

that corresponds to the only infinite root of equation (118).On the other hand, four other roots are finite, and (115) and (116) are suitable. In this case infinite roots of (118) have infinite values of a in correspondence, i.e., two osculating cylinders have turned into plane osculating curve L in point X. Curve has five infinitely close points with this plane, thus exist also conic in this plane, that has 4-fold touching with curve. All four osculating cylinders, which every two are symmetric in relation to osculating plane, go through this conic, because conic with each of them has common five points. From four osculating cylinders that don't coincide with the osculating plane, for real curves in general case two are imaginare; other two are imaginare, and either coincide with osculating plane or are real, depending from either conic is hyperbole, parabola or ellipse. In the case of a circle, all four osculating cylinders coincide with a cylinder whose circle is the exact intersection.

If we want at each point one of values ρ, ρ', τ to be zero, ρ =0 gives the trivial case of a straight line, τ=0 gives curves of plane for which at each point there exist osculating conic with tangency of order four and for which, thus, all conclusions of point c) are true. It remains only to consider interesting case ρ' = 0, i.e., transverse circle (šērsi rii ???) - curves with constant curvature. To be easier to characterize multiplicity of their coinciding osculating cylinders, let us determine for equations (112) to (114) equivalent equations in sufficiently general way to suit in other cases too, e.g., for isotropic curves.

In a similar way as before we could consider also cases of singular points X for which coefficients of some of equations (118) become infinite, that we are not to consider here.

3. Instead of Frenet formulas, we take the derivative formula

(121) $\vec{X}^{IV}$ = p$\vec{X}$' + q $\vec{X}$'' + r $\vec{X}$''' ,

that we may use for every point of spacial curve whereonly $\vec{X}$', $\vec{X}$'' and $\vec{X}$''' are not coplanar. Since corresponding case for curves where Frenet formulas are suitable is considered, we may assume that this condition is satisfied.

To express relatioin (108) in a handy way, we will define vector $\vec{Z}$ with relation

(122) $\vec{Z}$ =X - C - $\vec{u}$ [$\vec{u}$ (X-C)] .

Since $\vec{u}$ is vector of unity,

(123) $\vec{u}^2 = 1$

and vectors $\vec{u}$ and $\vec{Z}$ are perpendicular

(124) $\vec{u}\ \vec{Z} = 0$

Derivating relations where vector $\vec{Z}$ is contained, if $\vec{u}$ and C are constant, we are to set

(125) $\vec{Z}' = \vec{X}'$ - $\vec{u}$ ($\vec{u}$ X') .

Expressing equation (108) with the help of vector $\vec{Z}$, we get

(126) $\vec{Z}^2 = a^2$.

To characterize oscilating cylinder, this equation should be derivated four times, following conditions of derivation (125) and (121), taking a as constant. Derivating once, we get

$\vec{Z}\ \vec{X}'$ - $(\vec{u}\ \vec{Z})(\vec{u}\ \vec{X}') = 0$.

We are to note that last memeber we may drop: (124) show that its value is zero, and (125) , that by its derivation identically derivative of $\vec{u}\ \vec{Z}$ vanish, thus, derivatives of all members contain factor $\vec{u}\ \vec{Z}$ too, and are equal to zero. Derivating four times, we get

$$(127)\quad \begin{gathered} \vec{Z}\vec{X}' = 0 \\ \vec{Z}\vec{X}'' - \left(\vec{u}\vec{X}'\right)^2 + \vec{X}'^2 = 0 \\ \vec{Z}\vec{X}''' - 3\left(\vec{u}\vec{X}'\right)\left(\vec{u}\vec{X}''\right) + 3\vec{X}'\vec{X}'' = 0 \\ \vec{Z}\left(p\vec{X}' + q\vec{X}'' + r\vec{X}'''\right) - 4\left(\vec{u}\vec{X}'\right)\left(\vec{u}\vec{X}'''\right) - 3\left(\vec{u}\vec{X}''\right)^2 + 4\vec{X}'\vec{X}''' + 3\vec{X}'^2 = 0 \end{gathered}$$

First three equations uniquely determine vector $\vec{Z}$, if $\vec{u}$ is known, because according assumption $\vec{X}'$, $\vec{X}''$, $\vec{X}'''$ are not colinear. Equation (126), knowing $\vec{Z}$, gives a. To determine $\vec{u}$, we will use equation (123) and both equations that we obtain by exclusion of Z from (124) and (127). Introducing

$\vec{u}$ = x $\vec{X}'$ + y $\vec{X}''$ + x $\vec{X}'''$,

exclusion of $\vec{Z}$ from (124) to (127) we may perform by adding these equations, previously multiplying them correspondingly by: 1) +1, -x, -y, -z, 0 and 2) 0, p, q, r-1. Together with (123), obtained equations for determination of $\vec{u}$ give system

$$
(128)\quad
\begin{aligned}
&\left(\vec{u} = x\vec{X}' + y\vec{X}'' + x\vec{X}'''\right)^2 = 1\\
&3z\left(\vec{u}\vec{X}'\right)\left(\vec{u}\vec{X}''\right) + y\left(\vec{u}\vec{X}'\right)^2 - 3z\vec{X}'\vec{X}'' - y\vec{X}'^2 = 0\\
&4\left(\vec{u}\vec{X}'\right)\left(\vec{u}\vec{X}'''\right) + 3\left(\vec{u}\vec{X}''\right)^2 - 3r\left(\vec{u}\tilde{X}'\right)\left(\vec{u}\vec{X}''\right) - q\left(\vec{u}\vec{X}'\right)^2 -\\
&-4\vec{X}'\vec{X}''' - 3\vec{X}''^2 + 3r\vec{X}'\vec{X}'' + q\vec{X}'^2 = 0
\end{aligned}
$$

.

This system, for which equations are correspondingly second, third and fourth order with respect to components of $\vec{u}$, determine 12 vectors $\vec{u}$, which every two are opposite. Since two opposite vectors $\vec{u}$ determine the same cylinder, we see that in all cases, when formula is suitable, in general there exist six osculating cylinders. The seemingly cases of exception, when for system (128) not all systems of solutions give finite values of components of $\vec{u}$, may be interpreted as cases of degenerate cylinder. In any case, by multiplication of first equation (128) with handy factor, and adding it to both other, we obtain two homogeneous equations, that always should determine six relational systems of x, y and z.

In case of transversal circles we may choose unit of length (or do homothety) in a way that $\rho = 1$. After that, dropping simple case of circle $\tau = 0$,

$$
\begin{aligned}
\vec{X}' &= \vec{t}\\
\vec{X}'' &= \vec{n}\\
\vec{X}''' &= -\vec{t} + \tau\vec{b}\\
\vec{X}^{IV} &= -(1+\tau)^2\vec{n} + \tau'\vec{b}
\end{aligned}
$$

so that q = -1 -τ^2 , r= $\frac{\tau'}{\tau}$. Putting also x = b + z, (128) for determination of b, y and z, making in the mentioned way both last equations homogenous, we obtain

$$
(129)\quad \left\{
\begin{aligned}
&b^2 + y^2 + \tau^2 z^2 = 1\\
&y\left[3bz - y^2 - \tau^2 z^2\right] = 0\\
&-3b^2 + y^2\left(3 - \tau^2\right) - 3\tfrac{\tau'}{\tau}by + 4bz\tau^2 - \tau^4 z^2 = 0
\end{aligned}\right. .
$$

Both ways, by which we may satisfy second equation, correspond to already in end of page 78 mentioned cases. Both last equations (129) are satisfiet, if

$$
(130)\quad \left\{
\begin{aligned}
&y = 0\\
&\left(3b - \tau^2 z\right)\left(b - \tau^2 z\right) = 0
\end{aligned}\right.
$$

or else

$$
(131)\quad \left\{
\begin{aligned}
&y^2 = 3bz - \tau^2 z^2\\
&3\tfrac{\tau'}{\tau}by + \left(3b - \tau^2 z\right)(b - 3z) = 0
\end{aligned}\right. ,
$$

That shows that

$$
(132)\quad \left\{
\begin{aligned}
&y = 0\\
&3b - \tau^2 z = 0
\end{aligned}\right.
$$

is double solution. For these values of b, y, z, partial derivatives of both first equations (129) are proportional - we have come to case, when condition for matrix of values of partial equations are not satisfied. Because of this corresponding cylinder in general case is not cylinder of osculation. It can be seen, repeatingly derivating some of 4th order conditions of osculation, e.g., (114), where we put ρ'= 0 and obey (111). Values

$$(133) \quad \{ \quad \begin{array}{l} a = \frac{9\rho}{9\rho^2+\tau^2} \\ \phi = 0 \\ \mathrm{tg}\psi = \frac{3\rho}{\tau} \end{array} \quad ,$$

that correspond to solution (132), if ρ is arbrary and constant, satisfy 5th order touching condition (where $\phi = 0$ is already set)

(134) ρ τ'[-3 a τ + 5 sin ψ cos ψ] = 0

only if τ' = 0 (assuming $\rho \neq 0$). (133) characterized cylinder, thus, superosculate curve L only at points where τ' = 0. If we wish that superosculation occur in each point of L, τ too must be constant, so that curve should be simple srew line.

If curve L is simple srew line, cylinder on which it sits, and which is characterized by

$$(135) \quad \begin{array}{l} a = \frac{\rho}{\rho^2+\tau^2} \\ \phi = 0 \\ \mathrm{tg}\psi = \frac{\rho}{\tau} \end{array}$$

is encountered among solutions of (132) only once, while cylinder, that has (133) in correspondence, is 3-fold in general case, and 5-f$\leq$old, if τ = 3 ρ. Simple srew lines with $\tau = 3\rho$ is maximal order envelope of cylinders; they have supeosculation order 3, not 4, as was expectable according number of coinciding cylinders due to already mentioned property of partial derivatives.

To find out exact order of tangency and to check obtained results, let us take simple srew line L on cylinder with radius l. Denoting by x, y, z current coordinates of curve in non moving ortogonal system of coordinates, we may take

$$\begin{array}{l} x = \cos t \\ y = \sin t \\ z = 3t \end{array} \quad .$$

For this curve $\rho = \frac{1}{10}$, $\tau = \frac{3}{10}$ and (133) gives a = 5, tg ψ =1. Corresponding cylinder, that osculate L at origin point, has equation

40 (x-1) + 5 z^2 - (y - 2 z)^2 = 0 .

Insterting in left side of this equation values of current coordinates of L and expanding in power series, we obtain

40(cos t-1)+45 t^2- (sin t -6 t)^2 = 9/7!t^8 + ...

Curve L has with cylinder common 8 infinitemally close points and not 5, as in general case, thus, order of superosculation is exactly 3.

4. Let us use formula (128) in case of minimal curves. For theses curves Frenet fomulae are replaced by[18]

$$\vec{X}' = \vec{I}_1$$
$$\vec{I}_1' = \vec{I}_2$$
$$\vec{I}_2' = k\vec{I}_1 - \vec{I}_3$$
$$\vec{I}_3 = k\vec{I}_2$$

where
$$\vec{I}_1\ \vec{I}_3 = 1 \quad , \quad \vec{I}_2^{\,2} = 1$$

and remaing squares and scalar multiplications of these vectors are equal to zero. With help of these formulas, we obtain

p = k' , q = 2 k , r = 0

Inserting in system (128) and adding first equation to last, we obtain

$$-2xz + y^2 - 2kz^2 = 1$$
$$yz^2 = 0$$
$$2xz + 4y^2 + 4kz^2 = 0$$

Both last equations give: five times, y = z = 0 and once y = x + 2 k z = 0. The first system of values expresses that vector $\vec{u}$ is collinear with isotropic vector $\vec{X}$'. It may, thus, not have length 1; however, to obtain interpretation for such vector, we may imagine that isotropic curve occured by deforming some curve that were not isotropic, and for this curve to be deformed in corresponding equation of cylinder (108) to replace $\vec{u}$ with $\frac{\vec{u}}{d}$, where d is length of vector $\vec{u}$, that passes along axis of cylinder. Finally, (108) with d^2 and in boundary case, when as $\vec{u}$ may be taken $\vec{X}$', putting d=0, we obtain as equation of 5-fold degenerated cylinder

[$\vec{X}$'(X - C) $]^2$ = 0 ,

that characterize taken two fold osculating plane of curve. This interpretation is accordance with number of common infinitisemally close points of two-fold plane and curve, that is 6, thus, even by one more than oscilating cylinder should have.

For remaining the only non-degenerate cylinder

$\vec{u}$ = z[-2 k $\vec{X}$' +$\vec{X}$'''] , where $z^2 = \frac{1}{2k}$

and $\vec{Z} = z^2\ \vec{X}$'',

thus, a = $z^2 = \frac{1}{2k}$.

The radius of cylinder, thus, is equal to half from inverse quantity of lower differentialinvariant k. This interpretation for invariant gives without proof already Scheffers[19], only his utilized differentialinvariant is 16 k and, due to mistake or overwrite, is written as 3-fold ("vierpunktig"), and not 4-order tangency ???.

E. Cartan's[20] finds out that minimal curves with unchanging k≠0 are minimal simple srew lines; for arbitrary minimal curve L at each of its point X, where k≠0, exist such skrew line that touch it with order 4; this skrew line has the same curvature than curve L at point X.

E. Cartan's first result may be obtained also from our equations, expressing that (when?) occur superosculation, that gives k'=0. Since degenerated cylinder is single then superosculation at each point may occur onle when curve L sits on cylinder. E. Cartan's considered osculating srew line sits on osculating cylinder.

5. From results obtained in this paragraph Tameri consider only curves where Frenet formulae work and ignores imaginary cylinders, on p. 76 of cited work he mentions: that exist at most six osculating cylinders, not giving, however, explicit equations for their determination; that, in case ρ'=ρ''=0 there exist 2 to 5 osculating cylinders; that, if τ'=0 then condition is necessary to occur superosculation, that is sufficient for simple srew lines, that have 2 to 4 osculating cylinders, including cylinder on which they sit.

§2. On curves for which oscillating cylinders obey specified conditions

1. As already noted on p. 77, osculating cylinder of spatial curve may be subdued by two given conditions, because five functions of arc s a, ϕ, ψ, ρ, τ, that characterize curve and its family of osculating cylinders, are determined only by three relations (112) to (114). Giving two conditions more, that link these quantities and their derivatives, we will receive in all five differential equations with five variables. If equations are not incompatible, their solutions in general case will be dependent from arbitrary constants. Cases, of course, are possible, when equations have common solutions that depend from arbitrary functions. Requiring, e.g., that axis of cylinder were tangent to geometric place of point C - projection of current point X of curve on axis of cylinder - obviously all curves and on some unchanging cylinder give solution.

Depending from given conditions, it could be handy to eliminate among (112) to (114) not a and ψ to obtain relation (118) between t, ρ, τ and derivatives of both two, but some other two quantities, or, at all, express a, ϕ, ψ, ρ, τ with help of some other aids. Let us consider one such reduction with help of which it is possible to reduce solution of considered problem to solution of one first order differentian equation along with following quadratures. Assuming that ρ $\tau \neq$0, we put

(136) $A = \frac{\rho'}{\rho\tau}$, B = -3 $\frac{\tau'}{\Box\tau^2}$, D = 3 $\frac{\rho}{\tau}$, E = $\frac{5\rho'^2-3\rho\rho"}{\rho^2\tau^2}$

Then equation (118) transforms into

(137) $\left[t^4+Bt^3 + \text{E } t^2 + 2\text{A t} + A\hat{}2\right]\left(t^2+1\right) - D^2\, t^4 = 0$.

Quntities A, B, D, E are linked mutually with relation

(138) $(B-3A)\text{dA} = \left(\text{AB} + E - 2A^2\right)\frac{\text{dD}}{D}$

If A, B, D, E are obtained as one parameter functions, that satisfy this equation, value of curvature, twist and arc s we obtain with two quadratures, because

$$\frac{\text{d}\rho}{\rho} = \frac{3A\text{dD}}{(3A-B)D} = \frac{3A\text{dA}}{-2A^2+\text{AB}+E} ,$$

that give curvature. Twist is given by

$$\tau = 3\,\frac{\rho}{D}$$

and arc by

$$\text{ds} = \frac{\text{dD}}{\rho(B-3A)} = \frac{\rho\text{dD}-D\text{d}\rho}{\rho^2 B} .$$

Both expressions $\frac{\text{d}\rho}{\rho}$ and ds give the same values.

Arbitrary constant, that arise by determination of ρ, is constant of dimension - not knowing it, we obtain curves that are mutually similar. Constant that arise by determining s determine only point of the curve from where we start to count arcs.

Let us note yet values of quantities A, B, D, E for special curves. For general screw line D is constant and B=3A; to determine ρ and s we are to take last expressions of differentials, because first are indetermined. Cylindrical screw lines have four quantities as constants and besides

(139) B=3A, $E=-A^2$.

Previous formulas for determination of ρ and s, of course, are not suitable here, but if B and A are given, with one quadrature we obtain τ and ρ as functions of s. Finaly , simple screw lines have arbitrary constants, A=B= E = 0.

2. Let us find curve L for which exist three-fold osculating cylinder. Expressing that t is three-fold root of equation (137), and eliminating every two from obtained three equations quantities B, D, E , we obtain

$$(140)\quad \begin{aligned} B\left(3t^5-t^3\right) &= -8A^2+2A\left(t^3-3t\right)-8t^6 \\ D^2\left(3t^6-t^4\right) &= -\left(t^2+1\right)^3\left(3A^2+2\mathrm{At}+t^4\right) \\ B\left(3t^4-t^2\right) &= A^2\left(-3t^4-9t^2+6\right)+2A\left(-t^5-6t^3+3t\right)-t^8+3t^6 \end{aligned}$$

Let us first find out whether among solutions of problem is not general screw lines. They, as noticed, D is constant and B=3. Inserting latter condition in first equation (140), we obtain

$$8A^2+A\left(9t^5-5t^3+6t\right)+8t^6=0\ .$$

With equation (140) we have two equations with constant coefficients that link values of A and t. It may be checked that these equations are independent; they, thus, constants A and t. But then curve is cylindrical screw line or simple screw line. Since for last our problem is already solved, remains there case of cylindrical screw line. Then condition $E=-A^2$ gives second equation only linking A and t

$$A^2\left(-10t^2+6\right)+2A\left(-t^5-6t^3+3t\right)-t^8+3t^6=0\ .$$

Excluding A from both equations, we obtain

$$10t^4\left(3t^2-1\right)^2\left(t^2+1\right)\left(t^2-2\right)\left(t^4-t^2\text{-}2\right)=0$$

Root t=0 gives already considered case ρ'=0. $3t^2$-1=0 gives case, when (140), though, are satisfied, but these conditions are not equivalent with condition of existence of three-fold root, what are not satisfied all. t^2+1=0 give ρ=0. The remaining six t values give imaginare cylindrical srew lines, that are solution of the problem.

The rest lines for which at each point exist three-fold osculating cylinder, we get assuming that $t\left(3t^2\text{-}1\right)\neq 0$, expressing B, D, E with formula (140) as functions of A and t, and inserting obtained expressions into equation (13*), that gives

$$(141)\quad \begin{aligned} &t^2\left(t^2+1\right)\left(3t^2-1\right)M\mathrm{dA}+ \\ &+3\left(2A-t^3+t\right)\left[A\left(5t^2-1\right)+2t^5\right]N\mathrm{dt}=0 \end{aligned}$$

where

$$\begin{aligned} N=&A^3\left(-30t^2+26\right)+A^2\left(3t^5-43t^3+30t\right)+ \\ &+A\left(6t^8+10t^6-6t^4+6t^2\right)+7t^9+3t^7 \end{aligned}$$

$N=8A^3+A^2\left(9t^5+5t^3\right)+A\left(10t^6+12t^4-6t^2\right)+t^9-3t^7.$

Integrals of (141) for which dA≠0, dt≠0 give curves, that are not screw lines and for which at each point exist three-fold osculating cylinder with order of superosculation 2. We didn't succed in finding such integrals. Solution that are given by equating M dA to zero where just considered.

In analogoues way as before, we may seek curves L for which at each point exist two two-fold cylinders. Here A, B, D, E may be expressed as correspondign functions of t for both cylinders. Among solutions one more both cylindrical screw lines, and curves, that are not screw lines, and may be determined by on first order and first order algebraic differential equation.

We note yet that requirement for three roots of equation (137) to be constant characterize simple and cylindrical screw line for which roots are constants. For cylindrical screw lines eqution (137), it putting down, obtain form

$t^6+3\mathrm{At}^5+\left(1-A^2-D^2\right)t^4+5\mathrm{At}^3+2\mathrm{At}+A^2=0$.

Since quadratic memebers are absent, as well as coefficients of t^3 and t, if A and A are real, h

have the same signs, we conclude that at least one pair of roots is imaginare. Thus, for real cyclical screw linews at least two osculating cylinders are imaginare.

3. If the given conditions link quantities a, ϕ, ψ, it is handy to take the same as basic variables in order easy to interpret obtained relations. To simplify expressions we take denotations

$p=\sin\phi \; x=\sin\phi$

$q=\cos\phi \; y=\cos\psi$

The conditions of immobility (111) can be written as

$$\left\{\begin{array}{l} a'=0 \\ \psi'=\frac{\mathrm{px}^2}{aq} \\ \phi'=-\tau+\frac{xy}{a} \end{array}\right.$$

The conditions that actually characterize the immobility of the cylinder are the first two. The expression ϕ' is a consequence of these two due to the existing relations (112) and (113).

The quantities a, ϕ, ψ and their derivatives agains arc of the curve corresponding to one osculating cylinder are linked by relations

$$\frac{a'^2q}{ap^2}+a'\left[-2\psi'\,\frac{qy}{p^2x}-\frac{\phi'}{p}+3\,\frac{pxy}{a}\right]+ \qquad (142)$$
$$+\left(\psi'-\frac{px^2}{aq}\right)-\frac{a'qy}{p^2x}+4\,\psi'\,\frac{aqy^2}{p^2x^2}+2\,\phi'\,\frac{ay}{px}+3p\left(x^2-y^2\right)=0 \;.$$

This shows that the requirement that the radius a of a rotating cylinder be constant can be used in two ways: eighter using

(143) $\psi' - \frac{px^2}{aq}=0$,

then curve will sit/be on immovable/stationary cylinder, or else, putting

(144) $4\,\psi'\,\frac{aqy^2}{p^2x^2}+2\,\phi'\,\frac{ay}{px}+3\,p\left(x^2-y^2\right)=0$.

Curves for which this relation holds will not be on cylinder at all, because (144) and (143) obviously are not equivalent. By requiring for (143) and (144) to hold at the same time, we characterize curves of determined type which could be found by integrating differential equation

(145) 2c(1+c)db+(3bc+b+4)dc=0 ,

that arise between b = tg$^2\phi$ and c = tg$^2\psi$, eliminating arc s from relations (143) and (144). Integration and following determination of curve may be replace/postpone with geometric considerations. If for cylinder V on which curve L sits condition (144) holds, it may taken as limit state of some non-stationary/variable osculating cylinder with radius a_1 , that in limit has a_1= a and a_1'= 0. Curves L, thus, should be characteristic curves of some/separate surfaces V of family of one parameter cylinders with constant radius a.

Derivating equation (108) of general cylinder of such family, where as variables only C and $\vec{u}$ are taken, we obtain

(146) $\vec{C}$'(X-C)+[$\vec{u}$(X-C)][$\vec{u}$(X-C)-$\vec{u}$ $\vec{C}$] = 0 .

To characterize the curve L, determined by equation (108) and (146), as simple as possible, we take as the point C the center point of the axis of cylinder, considering this axis as generatrix of surface. Then

$\vec{C}$' $\vec{u}$' = 0 .

Taking orthogonal coordinate system with origin point C, axes x and z are passing correspondingly in directions of vectors u' and u, we express with help of parameter λ coordinates of current point of curve L in a way

X (a cos λ, a sin λ, k tg λ) ,

where k is component of $\vec{C}$' parallel to axis y. Computing here expressions of b and c corresponding to this curve and stationary cylinder, we find

b = $\frac{4k^2 \mathrm{tg}^2\lambda}{a^2\cos^4\lambda+k^2}$ c = $\frac{a^2}{k^2}$ cos$^4\lambda$.

Excluding λ, we obtain

c(b $c+b+4)^2 = 16\ \frac{a^2}{k^2}$ = const. ,

that is general integral of (145).

Simple curves are so called horotropic curves.

4. As the natural equation of curves on cylinder with radius a may be considered equation (143), or any of last two equatiions (111), if ϕ and ψ are ρ, ρ' and τ functions determined by equations (112) and (113). To express this equation only with ρ, τ and their derivatives, ϕ should be excluded from between equations (115) and (118) . It is not difficult to write down obtained equation, equating some determinant of order six to zero, but usefulness of this would be minor: to check whether any curve L is on a rotating cylinder, one should check whether the corresponding determinant is equal to zero or not. This attempt practically could be used, if curve L were given, because then sufices to find out that two sixth order equations against a corresponding to two points, that might be received by equting determinant to zero, don't have common roots, to find out that curve doesn't sit on cylinder; if, however, natural equation of curve L have any freely determinable parameters more, also this attempt to get negative answer is not practically useful. Because of this we will not mention this equation. Practically more useful, though long, would be such attempt: with help of relations (112) and (113), ρ and τ and their derivatives are expressed as

functions of constant a and variable s ϕ, ψ and their derivatives. Then should be checked whether relation that characterize given curve and (143) may be satisfied with such values of a, ϕ and ψ. If it is possible, and quantities are determined/expressed as one parameter functions, equation of curve in final form is obtainable with help of two quadratures, because angle between tangent and unit vector of stationary axis is known; if such values of a, ϕ and ψ don't exist, curve doesn't sit on cylinder.

Since oscilating cylinders we considered mainly to illustrate general result of first chapter, in order not to make this work too heavy, we will not consider also some other interesting features, that are determined by tangency of cylinder and curve, hoping to return to this in some future work.

1. G. Julia, Éléments de géométrie infinitésimale (Paris, Gauthier-Villars), 1927, 19. un20. l.p.

2. F. Klein, Vergleichende Betrachtungen über neuere geometrische Forschungen, Erlangen 1972, gedruckt Gesammelte Abhandlungen Bd. 1 (1921) S. 460-497. und Mathematische Annalen, Bd. 43 (1893).

3. See e.g. F. Klein, Vorlesungen über höhere Geometrie (Berlin, Springer), 1926, where these questions are considered repeatedly. Mentioned opinion of S. Lie see ibid, p.275

4. See e. g. F. Klein, loc. cit. 240. - 241. pp.

5. See e. g. H. G. Zeuthen, Lehrbuch der Abzahlenden Methoden der Geometrie (Leipzig, Teubner), 1914. 30.-31.pp.

6. J. Dubourdieu, Questions topologiques de géométrie différentielle. Mémor.d.Sc.Math. 78 (Paris, Gauthier-Villars) 1936, see 46-56 pp.

7. S. Lie, Classification und Integration von gewöhnlichen Differentialgleichungen, etc. III, Archiv for Math.og Naturw.9, 1883 371-458 pp, also Gesammelte Abhandlungen 5, 362-427 ppp, see first chapter

8. More detailed researches about cases of possible singular solutions and examples see: .. , (-), 1929,86-97, and 262-265 pp. G. Mammana, Sugli inviluppi di ordine superiore dei sistemi di curve piane. Giornale di Matematiche 65, 1927, 1.-20.pp.

9. Wide theory of movable reference system contain E. Cartan, La Theorie Des Groupes Finis et Continus La Geometrie Differentielle Traitees Par La Methode Du Repere Mobile. (Paris, Gauthier-Villars) 1937.

10. C. Darboux, Leçons sur la théorie générale des surfaces (Paris, GAuthier-Villers) 1887, I-IV, especially chapters I. and II.

11. E. Cesàro, Vorlesungen über Natürliche Geometrie (Leipzing, Teubner) 1927.

12. H. Piccioli, Sur un procédé pour parvenir à l'équation intrinsèque des lignes du cylindre de révolution, Nouvelles annales de mathématiques 4e série, tome 4 (1904), p. 402-405 Reference: Dr Salkowski, Fortschritte der Mathematik 35, 1904, 643p.

13. V. Hlavatý, Natürliche Gleichung der Kurven auf einer allgemeinen Fläche im metrischen Raume. Mathematische Zeitschrift 30, 1929, 470.-480.pp.

14. E. Cotton, Sur les courbes tracées sur une surface. Annales de la So. Polonaise de Mathematiques 17, 1938, 32.-41. pp. Reference: Fortschritte der

Mathematik 64, 1938, 711 pp.

15. loc.cit. 69.-74.pp. Equations (96) up to (98) are written using denotations of Cesaro

16. This interpretation is given already by Cesaro, loc.cit. 185.pp., mentioning that the problem: to find one parameter family of of spheres with given geometric place of centers and that osculate some curve, formulated Jamet.

17. A. Tamerl, Sitzungsberichte der Akademie der Wissenschaften in Wien, 140, 1931, 1.-10.pp.

18. E. Cartan, loc cit., see 34 pp.

19. G. Scheffers, Besondere transcendente Kurven, Enzykl.der Math.Wissenschaften III D 4, 1903,256.pp

20. loc.cit. 43p.